NELSON SENIOR MATHS METHODS 12
FULLY WORKED SOLUTIONS

Chapter 3 Applications of derivatives
Exercise 3.01 The increments formula

Concepts and techniques

1 Estimate $\sqrt{85}$

Let $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Let $x = 81$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{81}} = \frac{1}{18}$

$\delta x = 4$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= \frac{1}{18} \times 4 \\ &= \frac{2}{9}\end{aligned}$$

$$\sqrt{85} \approx 9 + \frac{2}{9} = 9\frac{2}{9}$$

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2 Estimate $\sqrt[3]{130}$

Let $y = \sqrt[3]{x} = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

Let x = 125, then $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{125^2}} = \frac{1}{3\times 5^2} = \frac{1}{75}$

$\delta x = 5$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times \delta x \\ &= \frac{1}{75}\times 5 \\ &= \frac{1}{15}\end{aligned}$$

$\sqrt[3]{130} \approx 5 + \frac{1}{15} = 5\frac{1}{15}$

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3 **a** Estimate $\sqrt{50}$

Let $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Let x = 49, then $\frac{dy}{dx} = \frac{1}{2\sqrt{49}} = \frac{1}{14}$

$\delta x = 1$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= \frac{1}{14} \times 1 \\ &= \frac{1}{14}\end{aligned}$$

$$\sqrt{50} \approx 7 + \frac{1}{14} = 7\frac{1}{14}$$

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b Estimate $\sqrt[4]{85}$

Let $y = \sqrt[4]{x} = x^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

Let x = 81, then $\frac{dy}{dx} = \frac{1}{4\sqrt[4]{81^3}} = \frac{1}{4\times 3^3} = \frac{1}{108}$

$\delta x = 4$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times\delta x\\ &= \frac{1}{108}\times 4\\ &= \frac{1}{27}\end{aligned}$$

$$\sqrt[4]{85} \approx 3 + \frac{1}{27} = 3\frac{1}{27}$$

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c Estimate $\sqrt{2536}$

Let $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Let x = 2500, then $\frac{dy}{dx} = \frac{1}{2\sqrt{2500}} = \frac{1}{100}$

$\delta x = 36$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= \frac{1}{100} \times 36 \\ &= 0.36\end{aligned}$$

$$\sqrt{2536} \approx 50 + 0.36 = 50\frac{9}{25} = 50.36$$

d Estimate $\sqrt[5]{250}$

Let $y = \sqrt[5]{x} = x^{\frac{1}{5}}$

$$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5\sqrt[5]{x^4}}$$

Let x = 243 (= 3^5), then $\frac{dy}{dx} = \frac{1}{5\sqrt[5]{243^4}} = \frac{1}{5\times 3^4} = \frac{1}{405}$

$\delta x = 7$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times\delta x\\ &= \frac{1}{405}\times 7\\ &= \frac{7}{405}\end{aligned}$$

$$\sqrt[5]{250} \approx 3+\frac{7}{405} = 3\frac{7}{405}$$

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e Estimate $\sqrt[6]{70}$

Let $y = \sqrt[6]{x} = x^{\frac{1}{6}}$

$$\frac{dy}{dx} = \frac{1}{6}x^{-\frac{5}{6}} = \frac{1}{6\sqrt[6]{x^5}}$$

Let $x = 64$ ($= 2^6$), then $\frac{dy}{dx} = \frac{1}{6\sqrt[6]{64^5}} = \frac{1}{6\times 2^5} = \frac{1}{192}$

$\delta x = 6$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times \delta x \\ &= \frac{1}{192}\times 6 \\ &= \frac{1}{32}\end{aligned}$$

$$\sqrt[6]{70} \approx 2 + \frac{1}{32} = 2\frac{1}{32}$$

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4 $64^{\frac{2}{3}} = \left(4^3\right)^{\frac{2}{3}} = 4^2 = 16$

Find an approximation for $67^{\frac{2}{3}}$.

Let $y = \sqrt[3]{x^2} = x^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

Let x = 64, then $\frac{dy}{dx} = \frac{2}{3\sqrt[3]{64}} = \frac{2}{3\times 4} = \frac{1}{6}$

$\delta x = 3$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times\delta x\\ &= \frac{1}{6}\times 3\\ &= \frac{1}{2}\end{aligned}$$

$\sqrt[3]{130} \approx 16 + \frac{1}{2} = 16\frac{1}{2}$

5 Find an approximation for 10.06^7.

Let $y = x^7$

$$\frac{dy}{dx} = 7x^6$$

Let x = 10, then $\frac{dy}{dx} = 7\times10^6$

$\delta x = 0.06$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times\delta x\\ &= 7\times10^6\times0.06\\ &= 42\times10^4\end{aligned}$$

$10.06^7 \approx 10^7 + 42\times10^4 = 10\ 420\ 000$

6 Find an approximation for 4.05^4.

Let $y = x^4$

$$\frac{dy}{dx} = 4x^3$$

Let $x = 4$, then $\frac{dy}{dx} = 4(4)^3 = 256$

$\delta x = 0.05$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= 256 \times 0.05 \\ &= 12.8\end{aligned}$$

$4.05^4 \approx (4)^4 + 12.8 = 256 + 12.8 = 268.8$

(compared to $4.05^4 = 269.042\ 006\ 25$ exactly)

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Reasoning and communication

7 $C(x)=4000+2.1x+0.01x^{\frac{3}{2}}$ and 5000 to 5100 racks per week.

$\delta C \approx C'(x)\times\delta x$ where $\delta x = 100$

$$C'(x)=2.1+\frac{3}{2}\times 0.01x^{\frac{1}{2}}$$
$$=2.1+0.015\sqrt{x}$$

At $x = 5000$, $C'(5000)=2.1+0.015\sqrt{5000}$

$$\delta C \approx \left(2.1+0.015\sqrt{5000}\right)\times 100$$
$$=316.066$$

The change in the production cost is approximately \$316.01.

8 $V=\frac{4}{3}\pi r^3$, $r = 5$ m, $\delta r = 0.1$ m, $\delta V = ?$

$$V_5=\frac{4}{3}\pi(5)^3$$

$\frac{dV}{dr}=4\pi r^2$ at $r = 5$ m, $\frac{dV}{dr}=100\pi$

$$\delta y \approx 100\pi\times 0.1$$
$$=10\pi$$

The percentage error of the calculated volume of the sphere

$$\approx \frac{\delta \text{V}}{\text{V}}\times 100\% = \frac{10\pi}{\frac{4}{3}\pi(5)^3}\times 100\% = 6\%$$

9 $y = 2x^3 - 3x^2 + 4x - 1$

a $\frac{dy}{dx} = 6x^2 - 6x + 4$

b At $x = 3$, $\frac{dy}{dx} = 40$

c δy = ?, as x increases from $x = 3$ to $x = 3.02$

$\delta x = 0.02$ at $x = 3$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= 0.02 \times 40 \\ &= 0.8\end{aligned}$$

10 $y = 4x^3 - 3x$, find the approximate increase in y as x increases from 2 to 2.03.

$\frac{dy}{dx} = 12x^2 - 3$

At $x = 2$, $\frac{dy}{dx} = 45$

δy = ?, as x increases from $x = 2$ to $x = 2.03$

$\delta x = 0.03$ at $x = 2$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= 0.03 \times 45 \\ &= 1.35\end{aligned}$$

11 $V=\frac{4}{3}\pi r^3$, r = 12 cm, δr = 0.05 cm, δV = ?

$\frac{dV}{dr}=4\pi r^2$ at r = 12 cm, $\frac{dV}{dr}=576\pi$

$$\delta V\approx\frac{dV}{dr}\times\delta\text{r}$$
$$=576\pi\times0.05$$
$$=90.48\ \text{cm}^3$$

12 **a** V = 375 mL = 375 cm^3

Let x = height

$$V=(\pi r^2)x$$
$$375=\pi(3)^2x$$
$$x=13.26\ \text{cm}$$

b $V=9\pi x$ at r = 3 cm

$$\frac{dV}{dx}=9\pi$$

$\delta V\approx\frac{dV}{dx}\times\delta x$ where small change in height is δx

$$\delta V\approx9\pi\times\delta x$$
$$=28.27\times\delta x\,\text{mL}$$

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c $V=\pi\left(\frac{y}{2}\right)^2\times x=\frac{y^2}{4}\pi\times 13.26$

$$\frac{dV}{dy}=\frac{y}{2}\pi\times\frac{375}{9\pi}$$

$$\delta V\approx\frac{dV}{dy}\times\delta y \quad \text{where small change in diameter is } \delta y$$

$$\approx\frac{6}{2}\pi\times\frac{375}{9\pi}\times\delta y=125\ \delta y\,\text{mL}\ \ \text{at } d=6$$

d $\delta V\approx 9\pi\times\delta x$ mL at $\delta x=0.1$ cm

$$\approx 9\pi\times 0.1\ \text{mL}=2.82\ \text{mL}$$

e $\delta V\approx\frac{d}{2}\pi x\times\delta d$ at $\delta d=0.2$ cm

$$\approx 125\times 0.2=25\ \text{mL}$$

13 **a** $g=\frac{4\pi^2 l}{T^2}=\frac{4(\pi)^2\times 2}{\left(\frac{57}{20}\right)^2}=9.72\ \text{m/sec}^2$

b $\frac{dg}{dT}=8\pi^2\left(-2T^{-3}\right)=\frac{-16\pi^2}{T^3}\ \ \text{at } l=2$

$$\frac{\delta g}{\delta t}=\frac{-16(\pi)^2}{\left(\frac{57}{20}\right)^3}\quad\text{where } T=\left(\frac{57}{20}\right)$$

$$\frac{\delta g}{\delta t}=-6.82$$

For one swing $\div 20$

$$\frac{\delta g}{\delta t}=-0.34$$

$$\delta g=-0.34\delta t$$

c If $\delta T = \pm 0.5$ s, $\delta g \approx -0.34 \times \pm\frac{1}{2} = \pm 0.17\,\text{m/s}$

The error in $g \approx 0.17\text{m/s}^2$.

14 $A = x^2$, $\delta x = 1$ mm

$\delta A \approx \frac{dA}{dx} \times \delta x$

$\delta A \approx (2x) \times \delta x$

$= 2(1000) \times 1\ \text{mm}^2$

$\delta A = 2000\ \text{mm}^2$

15 $P(n) = 2000n + 10n^2$, $n = 18$, $\delta n = 5\%$ of $80 = 4$

a $\delta P \approx P'(n) \times \delta n$ where $\delta n = 4$

$P'(n) = 2000 + 20n$

$P'(18) = 2000 + 20 \times 80 = 3600$

$\delta P \approx 3600 \times 4 = 14\,400$

b $\delta n = 8\%$ of $80 = 6.4$

$\delta P \approx 3600 \times 6.4 = 23\,040$

c $\delta n = 10\%$ of $80 = 8$

$\delta P \approx 3600 \times 8 = 28\,800$

16 $V = x^3$, $x = 17$ cm, $\delta x\% = 2\%$, $\delta V\% = ?$

$$\frac{dV}{dx} = 3x^2$$

At $x = 17$, $\frac{dV}{dx} = 867$

$$\begin{aligned}\delta V &\approx \frac{dV}{dx} \times \delta x \\ &= 867 \times \left(\frac{2}{100} \times 17\right) \\ &= 294.78\end{aligned}$$

Percentage error in V is $\frac{294.78}{17^3} \times 100\% = 6\%$.

17 $A = \pi r^2$

$$\begin{aligned}\delta A &\approx \frac{dA}{dr} \times \delta r \\ &= (2\pi r) \times \delta r\end{aligned}$$

$$\begin{aligned}\frac{\delta A}{A} \times 100 &\approx \frac{(2\pi r)}{A} \times \delta r \times 100 \\ &= \frac{(2\pi r)}{\pi r^2} \times \delta r \times 100 \\ &= 2 \times \frac{\delta r}{r} \times 100\end{aligned}$$

But $\frac{\delta A}{A} \times 100 = 2\%$

$$\cancel{2} = \cancel{2} \times \frac{\delta r}{r} \times 100$$

$$\frac{\delta r}{r} \times 100 = 1\%$$

Therefore the approximate maximum allowable percentage error that may be made in measuring the radius is 1%.

Exercise 3.02 The second derivative

Concepts and techniques

1 Let $y = x^7 - 2x^5 + x^4 - x - 3$

$$\frac{dy}{dx} = 7x^6 - 10x^4 + 4x^3 - 1$$
$$\frac{d^2y}{dx^2} = 42x^5 - 40x^3 + 12x^2$$

2 $f(x) = x^9 - 5$

$f'(x) = 9x^8$

$f''(x) = 72x^7$

3 $f(x) = 2x^5 - x^3 + 1$

$f'(x) = 10x^4 - 3x^2$

$f''(x) = 40x^3 - 6x$

4 $y = x^7 - 2x^5 + 4x^4 - 7$

$$\frac{dy}{dx} = 7x^6 - 10x^4 + 16x^3$$
$$\frac{d^2y}{dx^2} = 42x^5 - 40x^3 + 48x^2$$

5 $y = 5 \cos (2x)$

$$\frac{dy}{dx} = -10\sin(2x)$$
$$\frac{d^2y}{dx^2} = -20\cos(2x)$$

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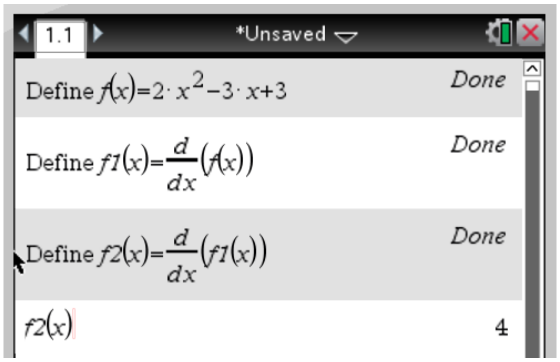
6 TI-Nspire CAS



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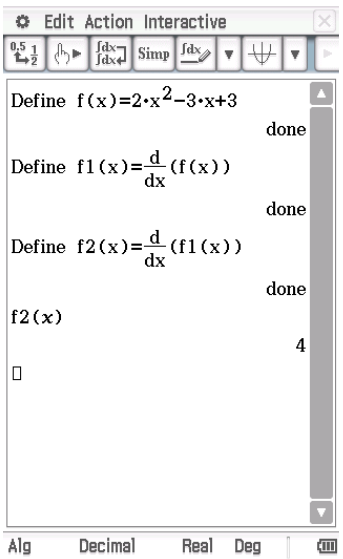
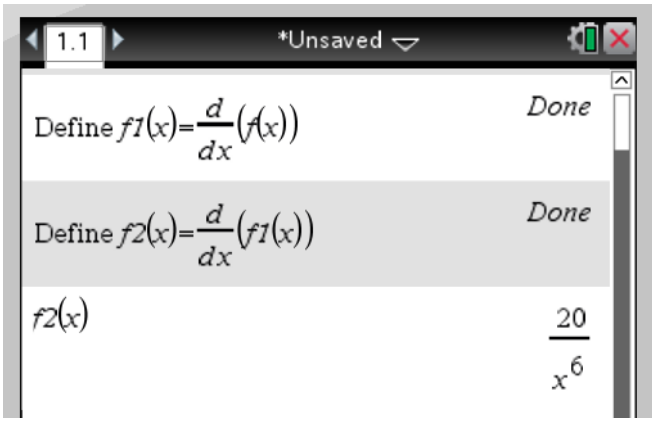


a $y = 2x^2 - 3x + 3$

$$\frac{dy}{dx} = 4x - 3$$

$$\frac{d^2y}{dx^2} = 4$$

b TI-Nspire CAS



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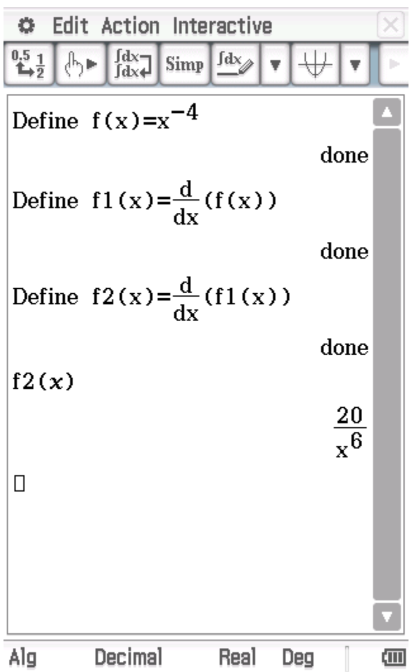

$y = x^{-4}$

$$\frac{dy}{dx} = -4x^{-5}$$

$$\frac{d^2y}{dx^2} = 20x^{-6}$$

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7 $f(t) = 3t^4 - 2t^3 + 5t - 4$

$f'(t) = 12t^3 - 6t^2 + 5$

$f''(x) = 36t^2 - 12t$

$f'(1) = 11$

$f''(-2) = 168$

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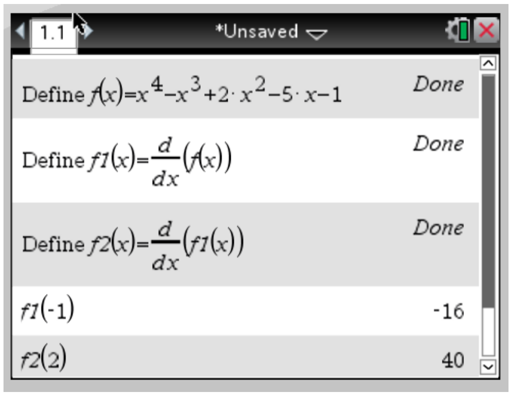
8 **TI-Nspire CAS**



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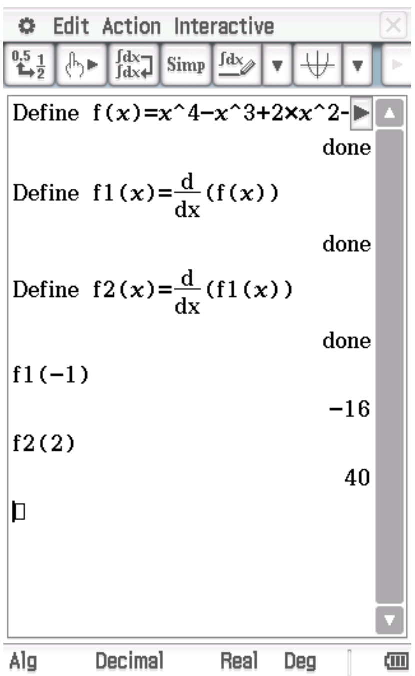

$f(x) = x^4 - x^3 + 2x^2 - 5x - 1$

$f'(x) = 4x^3 - 3x^2 + 4x - 5$

$f''(x) = 12x^2 - 6x + 4$

$f'(-1) = -16$

$f''(2) = 40$

9 $g(x) = \sqrt{x}$

$$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$g''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \frac{-1}{4\sqrt{x^3}}$$

$$g''(4) = \frac{-1}{4\sqrt{4^3}} = -\frac{1}{32}$$

10 $h = 5t^3 - 2t^2 + t + 5$

$$\frac{dh}{dt} = 15t^2 - 4t + 1$$

$$\frac{d^2h}{dt^2} = 30t - 4$$

At $t = 1$

$$\frac{d^2h}{dt^2} = 26$$

11 $f(x) = \sqrt{2-x}$

$$f'(x) = \frac{1}{2}(2-x)^{-\frac{1}{2}}(-1) = \frac{-1}{2\sqrt{2-x}}$$

$$f''(x) = -\frac{1}{4}(2-x)^{-\frac{3}{2}}(-1)^2 = -\frac{1}{4\sqrt{(2-x)^3}}$$

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12 $f(x)=\dfrac{x+5}{3x-1}$

$$f'(x)=\frac{1\times\left(3x-1\right)-3\left(x+5\right)}{\left(3x-1\right)^2}=\frac{-16}{\left(3x-1\right)^2}$$

$$f''(x)=-16(-2)\left(3x-1\right)^{-3}\times 3=\frac{96}{\left(3x-1\right)^3}$$

13 $v = (t + 3)(2t - 1)^2$

$$\begin{aligned}\frac{dv}{dt}&=1\times(2t-1)^2+2(2t-1)\times 2\left(t+\ 3\right)\\&=(2t-1)\left(2t-1+4t+12\right)\\&=(2t-1)\left(6t+11\right)\end{aligned}$$

$$\begin{aligned}\frac{d^2v}{dt^2}&=2(6t+11)+6(2t-1)\\&=24t+16\end{aligned}$$

Reasoning and communication

14 $y = 3x^3 - 2x^2 + 5x$

$$\frac{dy}{dx}=9x^2-4x+\ 5$$

$$\frac{d^2y}{dx^2}=18x-4$$

If $\dfrac{d^2y}{dx^2}$ = 3, then $3 = 18x - 4 \Rightarrow x = \dfrac{7}{18}$

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15 $f(x) = 2x^3 - x^2 + x + 9$

$$f'(x) = 6x^2 - 2x + 1$$

$$f''(x) = 12x - 2$$

$f''(x) > 0$ for $x > \frac{1}{6}$

16 $y = [4 \sin(x) - 2]^5$

$$\frac{dy}{dx} = 5[4 \sin(x) - 2]^4 \left[4 \cos(x)\right] = 20 \cos(x)[4 \sin(x) - 2]^4$$

$$\frac{d^2y}{dx^2} = -20 \sin(x)[4 \sin(x) - 2]^4 + 80 \cos(x)[4 \sin(x) - 2]^3 \left[4 \cos(x)\right]$$

$$= 320 \cos^2(x)[4 \sin(x) - 2]^3 - 20 \sin(x)[4 \sin(x) - 2]^4$$

17 $f(x) = 2 \sin\left(\frac{x}{2}\right) - 3 \cos^2(x) + 1$

$$f'(x) = 2 \times \frac{1}{2} \cos\left(\frac{x}{2}\right) - 6 \cos(x)\left[-\sin(x)\right] = \cos\left(\frac{x}{2}\right) + 6 \sin(x) \cos(x) = \cos\left(\frac{x}{2}\right) + 3 \sin(2x)$$

$$f''(x) = \frac{1}{2}\left[-\sin\left(\frac{x}{2}\right)\right] + 6 \cos(x) \cos(x) - 6 \sin(x) \sin(x) = -\frac{1}{2} \sin\left(\frac{x}{2}\right) + 6 \cos^2(x) - 6 \sin^2(x)$$

$$= -\frac{1}{2} \sin\left(\frac{x}{2}\right) + 6 \cos(2x)$$

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18 $y = bx^3 - 2x^2 + 5x + 4$

$$\frac{dy}{dx} = 3bx^2 - 4x + 5$$
$$\frac{d^2y}{dx^2} = 6bx - 4$$

$\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$, so $3b - 4 = -2 \Rightarrow 3b = 2 \Rightarrow b = \frac{2}{3}$

19 $f(x) = 5bx^2 - 4x^3$

$f'(x) = 10bx - 12x^2$

$f''(x) = 10b - 24x$

$f''(-1) = -3$, so $-3 = 10b - 24(-1)$

$10b = -27$

$b = -2.7$

20 $x = t^3 - 7t + 4$

$v = 3t^2 - 7$

$a = 6t$

At $t = 3$, $v = 20$ m/s and $a = 18$ m/s^2

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21 $x(t) = t^3 - 6t^2 + 8t + 5$

$v(t) = 3t^2 - 12t + 8$

$a(t) = 6t - 12$

a $v\ (2) = -4$ m/s

b $v(4) = 8$ m/s

c $a(2) = 0$ m/s^2

d $a(5) = 18$ m/s^2

22 $d(t) = 7t^2 - 2t^3 + 3\text{t} + 3$

$v(t) = 14t - 6t^2 + 3$

$a(t) = 14 - 12t$

a $v(1) = 11$ m/s

b $v(3) = -9$ m/s

c $a(1) = 2$ m/s^2

d $a(3) = -22$ m/s^2

23 $x = t^3 + 6t^2 - 2t + 1$ m

a $v = 3t^2 + 12t - 2$

$a = 6t + 12$

b $x_5 = 266$ m

c $v_5 = 133$ m/s

d $a_5 = 42$ m/s^2

24 $s = ut + \frac{1}{2}gt^2$

a $v = u + gt$

i.e. $v = 2 - 10t$ m/s

b $v_{10} = -98$ m/s

c Using $v = u + gt$, $a = -10 = g$

25 $s = \dfrac{2t-5}{3t+1}$

$$\begin{aligned} v &= \frac{2(3t+1)-3(2t-5)}{(3t+1)^2} \\ &= \frac{17}{(3t+1)^2} \\ a &= -2\times 17(3t+1)^{-3}\times 3 \\ &= -\frac{102}{(3t+1)^3} \end{aligned}$$

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Exercise 3.03 The second derivative and concavity

Concepts and techniques

1 $y = x^3 + x^2 - 2x - 1$

$$\frac{dy}{dx} = 3x^2 + 2x - 2$$
$$\frac{d^2y}{dx^2} = 6x + 2$$

Concave upwards for $\frac{d^2y}{dx^2} > 0$, i.e. $6x + 2 > 0 \Rightarrow x > -\frac{1}{3}$

2 $y = (x-3)^3$

$$\frac{dy}{dx} = 3(x-3)^2 \times 1$$
$$\frac{d^2y}{dx^2} = 6(x-3)$$

Concave downwards for $\frac{d^2y}{dx^2} < 0$, i.e. $6(x-3) < 0 \Rightarrow x < 3$

3 $y = 8 - 6x - 4x^2$

$$\frac{dy}{dx} = -6 - 8x$$
$$\frac{d^2y}{dx^2} = -8 < 0 \text{ for all values of } x$$

Therefore $y = 8 - 6x - 4x^2$ is always concave downwards.

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4 $y = x^2$

$$\frac{dy}{dx} = 2x$$
$$\frac{d^2y}{dx^2} = 2 > 0 \text{ for all values of } x$$

Therefore $y = x^2$ is always concave upwards.

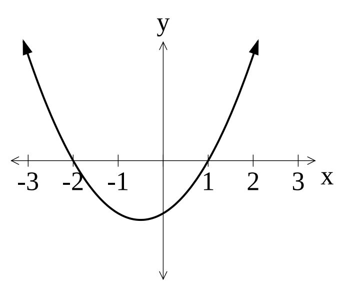
5 $f(x) = x^3 - 7x^2 + 1$

$$f'(x) = 3x^2 - 14x$$
$$f''(x) = 6x - 14$$

Concave downwards for $\frac{d^2y}{dx^2} < 0$, i.e. $6x - 14 < 0 \Rightarrow x < \frac{7}{3}$

6 $f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1$

$$f'(x) = 4x^3 + 6x^2 - 24x + 12$$
$$f''(x) = 12x^2 + 12x - 24$$



Concave downwards for $\frac{d^2y}{dx^2} < 0$

i.e. $12x^2 + 12x - 24 < 0$

i.e. $x^2 + x - 2 < 0$

$(x+2)(x-1) < 0$

i.e. $-2 < x < 1$

7 **a** $y = x^6$

$\frac{dy}{dx} = 6x^5$

$\frac{d^2y}{dx^2} = 30x^4$

Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

It does not change sign, so there is no point of inflection.

b $y = x^7$

$\frac{dy}{dx} = 7x^6$

$\frac{d^2y}{dx^2} = 42x^5$

Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

Yes, at (0, 0).

c $y = x^5$

$\frac{dy}{dx} = 5x^4$

$\frac{d^2y}{dx^2} = 20x^3$

Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

Yes, at (0, 0).

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d $y = x^9$

$$\frac{dy}{dx} = 9x^8$$
$$\frac{d^2y}{dx^2} = 72x^7$$

Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

Yes, at (0, 0).

e $y = x^{12}$

$$\frac{dy}{dx} = 12x^{11}$$
$$\frac{d^2y}{dx^2} = 132x^{10}$$

Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

It does not change sign, so there is no point of inflection.

8 Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

$g(x) = x^3 - 3x^2 + 2x + 9$.

$$g'(x) = 3x^2 - 6x + 2$$
$$g''(x) = 6x - 6$$

Point of inflection at (1, 9).

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9 Points of inflection occur where $\frac{d^2y}{dx^2} = 0$ and it changes sign.

Given $y = x^4 - 6x^2 + 12x - 24$

$$\frac{dy}{dx} = 4x^3 - 12x + 12$$

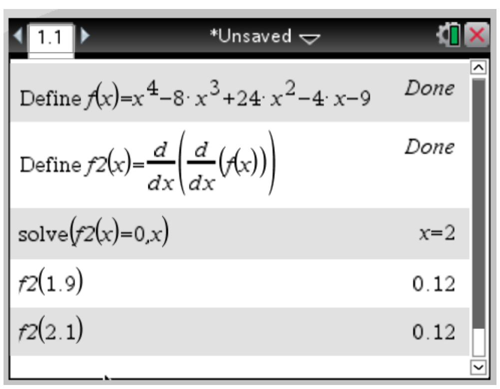
$$\frac{d^2y}{dx^2} = 12x^2 - 12 = 12(x^2 - 1)$$

$$\frac{d^2y}{dx^2} = 0 \text{ at } x = 1 \text{ or } x = -1$$

Points of inflection are (1, –17), and (–1, –41) (2nd derivative changes sign at both)

10

TI-Nspire CAS



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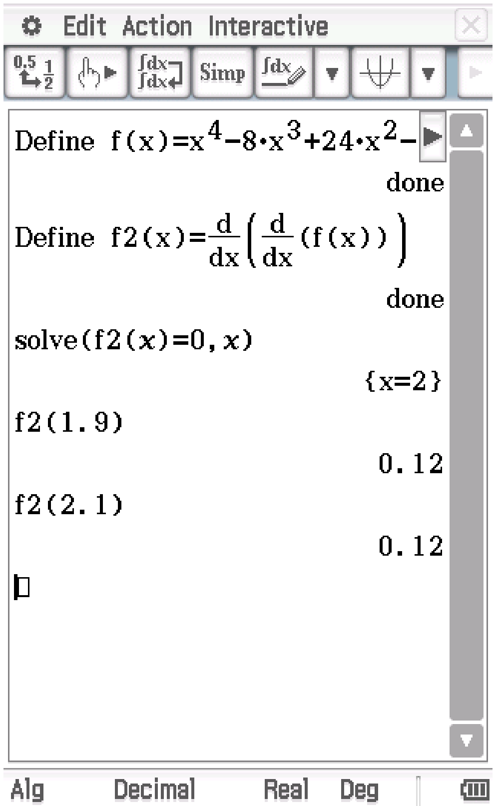


Points of inflection occur where $\frac{d^2y}{dx^2} = 0$.

Given $y = x^4 - 8x^3 + 24x^2 - 4x - 9$

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 48x - 4$$

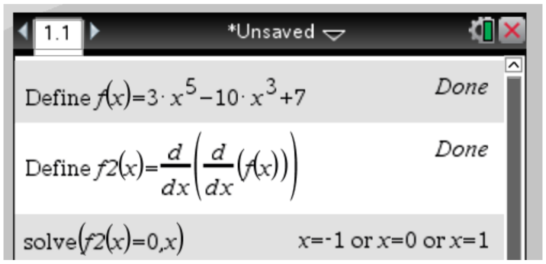
$$\frac{d^2y}{dx^2} = 12x^2 - 48x + 48 = 12(x-2)^2$$

Possibly a point of inflection at $x = 2$, but at $x = 2^+$ and at $x = 2^-$ the value of $\frac{d^2y}{dx^2}$

remains positive, so concavity does not change.

Therefore there is no point of inflection.

11 TI-Nspire CAS



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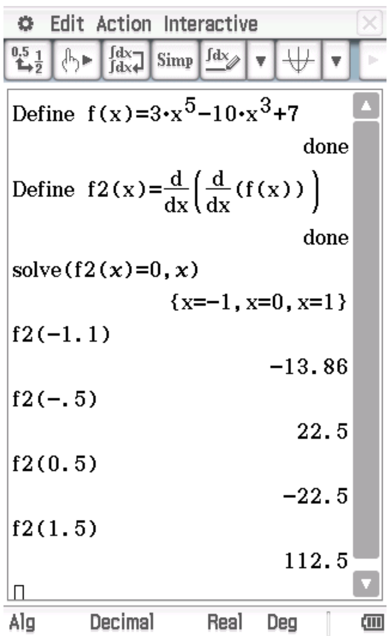

Points of inflection occur where $f''(x) = 0$

Given $f(x) = 3x^5 - 10x^3 + 7$

$f'(x) = 15x^4 - 30x^2$

$f'(x) = 60x^3 - 60x = 60x(x^2 - 1)$

$f''(x) = 0$ at $x = 0, \pm 1$

Points of inflection are $(0, 7), (1, 0)$ and $(-1, 14)$ $\left(\text{2nd derivative changes sign at all three}\right)$

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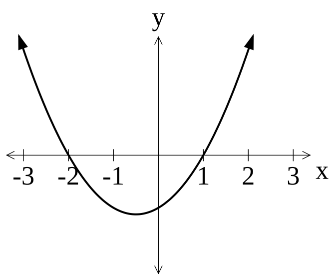
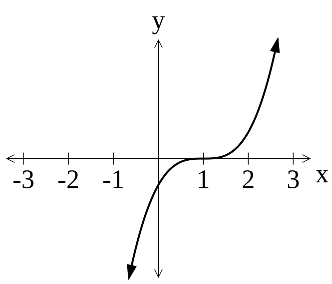
Reasoning and communication

12



Other answers are possible.

13



Other answers are possible.

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14 $f(x) = \frac{2}{x^2}$

$f'(x) = -\frac{4}{x^3}$

$f''(x) = \frac{12}{x^4}$

$\frac{12}{x^4} > 0$ for all $x \neq 0$

$\therefore \quad f''(x) > 0 \quad$ for $x \neq 0$

So concave up for $x \neq 0$

15 **a** $y = x^4 + 12x^2 - 20x + 3$

$\frac{dy}{dx} = 4x^3 + 24x - 20$

$\frac{d^2y}{dx^2} = 12x^2 + 24$

$12x^2 \geq 0$

$12x^2 + 24 > 0$

$\frac{d^2y}{dx^2} \neq 0$

Therefore the function has no points of inflection.

b $\frac{d^2y}{dx^2} > 0$ therefore the curve is concave upwards.

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16 $y = ax^3 - 12x^2 + 3x - 5$

$$\frac{dy}{dx} = 3ax^2 - 24x + 3$$
$$\frac{d^2y}{dx^2} = 6ax - 24$$
but $\frac{d^2y}{dx^2} = 0$ at $x = 2$
$$\Rightarrow 6a \times 2 - 24 = 0$$
$$a = 2$$

17 $f(x) = x^4 - 6px^2 - 20x + 11$

$$f'(x) = 4x^3 - 12px - 20$$
$$f''(x) = 12x^2 - 12p$$
but $f''(x) = 0$ at $x = -2$
$\Rightarrow 12x^2 - 12p = 0$ at $x = -2$
$$48 = 12p$$
$$p = 4$$

18 $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$

$$\frac{dy}{dx} = 8ax^3 + 12bx^2 - 144x + 4$$
$$\frac{d^2y}{dx^2} = 24ax^2 + 24bx - 144$$
but $\frac{d^2y}{dx^2} = 0$ at $x = 2$ and $x = -1$
$$24ax^2 + 24bx - 144 = 0$$
At $x = 2$: $96a + 48b - 144 = 0 \quad \Rightarrow 2a + b - 3 = 0$
At $x = -1$: $24a - 24b - 144 = 0 \quad \Rightarrow a - b - 6 = 0$
$b = 3 - 2a \quad \Rightarrow \quad a - 3 + 2a - 6 = 0$
$$a = 3, \quad b = -3$$

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Exercise 3.04 The second derivative test

Concepts and techniques

1 $y = x^2 - 2x + 1$

$\frac{dy}{dx} = 2x - 2$

$\frac{d^2y}{dx^2} = 2$

Turning point at $\frac{dy}{dx} = 0$ i.e. at $x = 1$

$\frac{d^2y}{dx^2} > 0$

Therefore the function is concave up, so (1, 0) is a local minimum turning point.

2 $y = 3x^4 + 1$

$\frac{dy}{dx} = 12x^3$

$\frac{d^2y}{dx^2} = 36x^2$

Turning point at $\frac{dy}{dx} = 0$ i.e. stationary at $x = 0$

$\frac{d^2y}{dx^2} = 0 \Rightarrow$ stationary point of inflection

x	–1	0	1
$\frac{d^2y}{dx^2}$	36	0	36

Concavity does not change, so concave up for all values of *x* except the turning point.

Minimum value at (0, 1).

3 $y = 3x^2 - 12x + 7$

$\dfrac{dy}{dx} = 6x - 12$

$\dfrac{d^2y}{dx^2} = 6$

Turning point at $\dfrac{dy}{dx} = 0$ i.e. at $x = 2$

$\dfrac{d^2y}{dx^2} > 0$

Therefore the function is concave up, so (2, –5) is a local minimum turning point.

4 $y = x - x^2$

$\dfrac{dy}{dx} = 1 - 2x$

$\dfrac{d^2y}{dx^2} = -2$

Turning point at $\dfrac{dy}{dx} = 0$ i.e. at $x = \dfrac{1}{2}$

$\dfrac{d^2y}{dx^2} < 0$

Therefore $f(x)$ is concave down, so $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$ is a local maximum turning point.

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5 $f\left(x\right)=2x^3-5$

$f'(x)=6x^2$

$f''(x)=12x$

Turning point at $f'(x)=0$ i.e. stationary at $x=0$

$f''(0)=0 \Rightarrow$ stationary point of inflection.

x	–1	0	1
$f''(x)$	–12	0	12

Concave down for $x < 0$ and concave up for $x > 0$.

Horizontal point of inflection at (0, –5).

6 $f\left(x\right)=3x^5+8$

$f'(x)=15x^4$

$f''(x)=60x^3$

Turning point at $f'(x)=0$ i.e. stationary at $x=0$

$f''(0)=0 \Rightarrow$ horizontal point of inflection.

x	–1	0	1
f''(x)	-60	0	60

Concave down for $x < 0$ and concave up for $x > 0$

Yes. Horizontal point of inflection at (0, 8).

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7 $f(x) = 2x^3 + 15x^2 + 36x - 50$

$f'(x) = 6x^2 + 30x + 36$

$f''(x) = 12x + 30$

Turning point at $f'(x) = 0$

i.e. $6x^2 + 30x + 36 = 0$

$x^2 + 5x + 6 = 0$

$(x+2)(x+3) = 0$

Stationary points at $x = -2, -3$

At $x = -2$, $f''(-2) = -24 + 30 > 0$ so minimum turning point at $(-2, -78)$.

At $x = -3$, $f''(-3) = -36 + 30 < 0$ so maximum turning point at $(-3, -77)$.

8 $y = 3x^4 - 4x^3 - 12x^2 + 1$

$\frac{dy}{dx} = 12x^3 - 12x^2 - 24x$

$\frac{d^2y}{dx^2} = 36x^2 - 24x - 24$

Turning point at $\frac{dy}{dx} = 0$

i.e. $12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$

Stationary points at $x = 0, 2, -1$

At $x = 0$, $\frac{d^2y}{dx^2} = -24 < 0$ so maximum turning point at (0, 1).

At $x = 2$, $\frac{d^2y}{dx^2} = 72 > 0$ so minimum turning point at $(2, -31)$.

At $x = -1$, $\frac{d^2y}{dx^2} = 36 > 0$ so minimum turning point at $(-1, -4)$.

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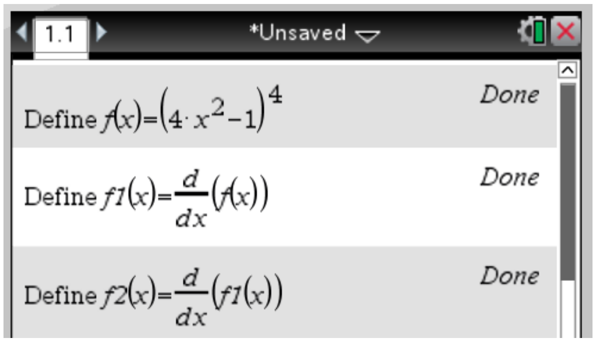
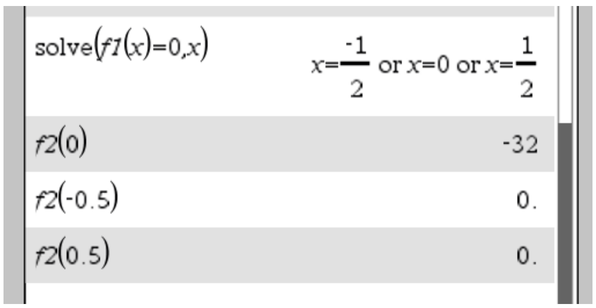
9 **TI-Nspire CAS**





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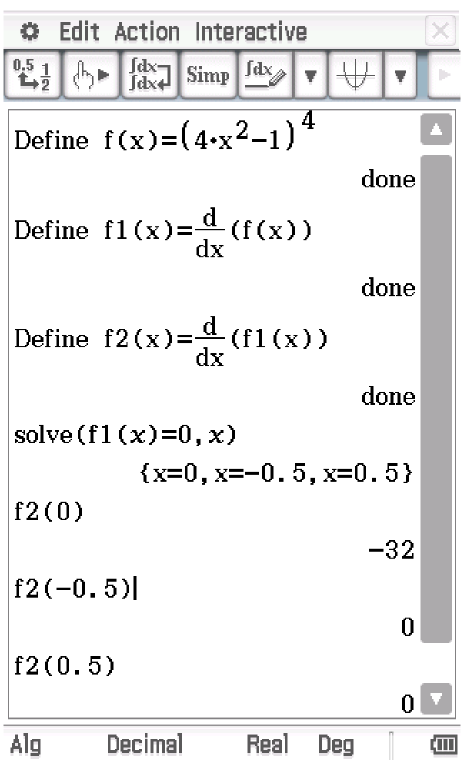

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$y=\left(4x^2-1\right)^4$

$$\frac{dy}{dx}=4\left(4x^2-1\right)^3\times 8x$$
$$=32x\left(4x^2-1\right)^3$$
$$\frac{d^2y}{dx^2}=32\left(4x^2-1\right)^3+3\left(4x^2-1\right)^2\times 8x\times 32x$$
$$=32\left(4x^2-1\right)^2\left(4x^2-1+24x^2\right)$$
$$=32\left(4x^2-1\right)^2\left(28x^2-1\right)$$

Turning point at $\frac{dy}{dx}=0$

i.e. $32x\left(4x^2-1\right)^3=0$

$x=0$ or $4x^2=1$

$x=\pm\frac{1}{2}$

At $x=0$, $\frac{d^2y}{dx^2}=32(-1)^2(-1)<0$, so maximum turning point at (0,1).

At $x=\pm\frac{1}{2}$, $\frac{d^2y}{dx^2}=32(1-1)^2\left[28\left(\pm\frac{1}{2}\right)^2-1\right]=0$ stationary point of inflection.

x	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1
$f'(x)$	+	0	+	+	0	+

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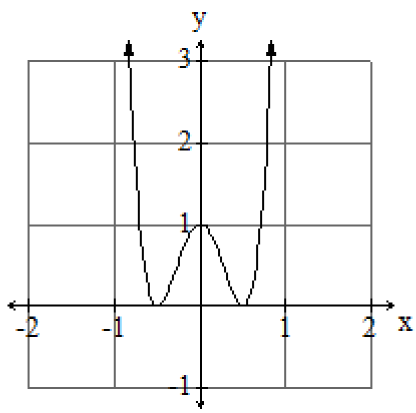
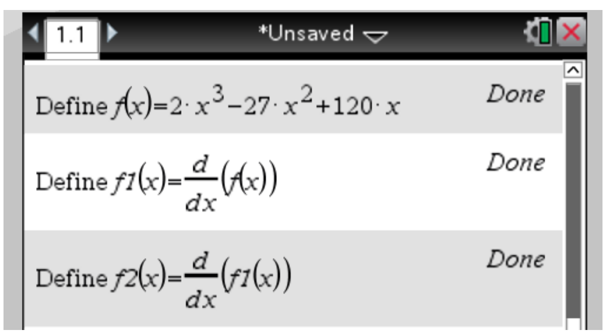
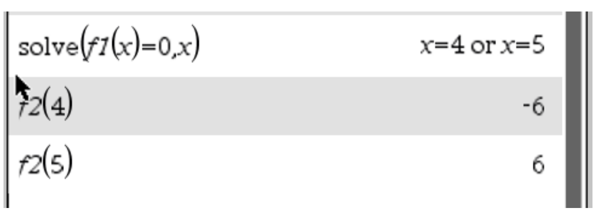
Concave up at $x = 0.5$ and at $x = –0.5$ so minimum turning points at (0.5, 0)

and at (–0.5, 0)



10 TI-Nspire CAS





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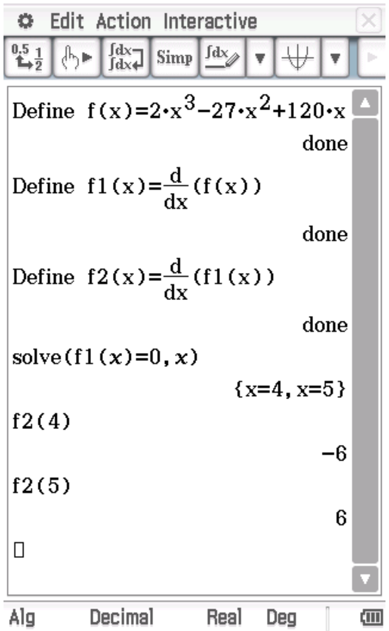


$y = 2x^3 - 27x^2 + 120x$

$\frac{dy}{dx} = 6x^2 - 54x + 120$

$\frac{d^2y}{dx^2} = 12x - 54$

Turning point at $\frac{dy}{dx} = 0$

$6x^2 - 54x + 120 = 0$

$x^2 - 9x + 20 = 0$

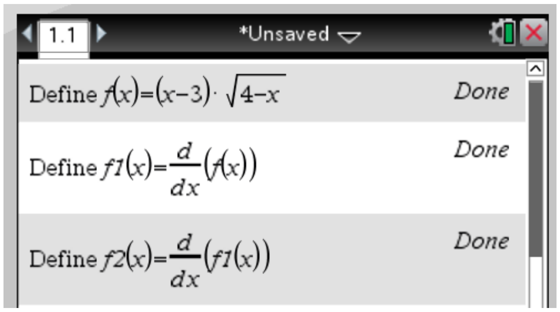
$(x-4)(x-5) = 0$

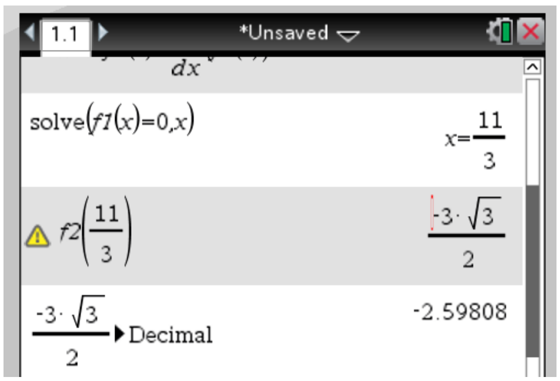
$x = 4$ or $x = 5$

At $x = 4$, $\frac{d^2y}{dx^2} = 16 - 54 < 0$, so maximum turning point at (4, 176).

At $x = 5$, $\frac{d^2y}{dx^2} = 60 - 54 > 0$ so minimum turning point at (5, 175).

11 TI-Nspire CAS





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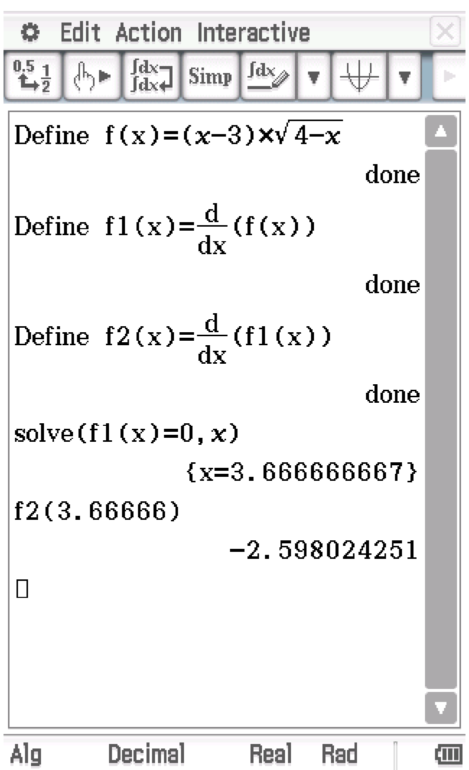

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$y = (x-3)\sqrt{4-x}$

$$\frac{dy}{dx} = 1\sqrt{4-x} + \frac{1}{2}(4-x)^{-\frac{1}{2}}(-1)(x-3)$$
$$= \frac{2(4-x)-(x-3)}{2\sqrt{4-x}}$$
$$= \frac{11-3x}{2\sqrt{4-x}}$$

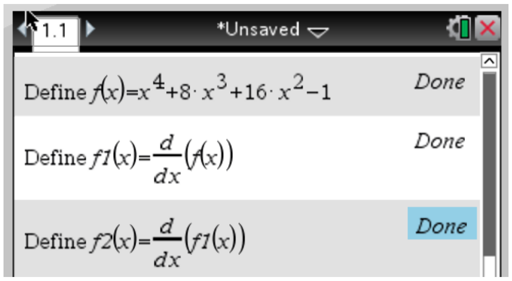
$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{-3\sqrt{4-x} - \frac{1}{2}(4-x)^{-\frac{1}{2}}(-1)(11-3x)}{4-x}\right]$$
$$= \frac{1}{2(4-x)}\left[-3\sqrt{4-x} + \frac{(11-3x)}{2\sqrt{4-x}}\right]$$

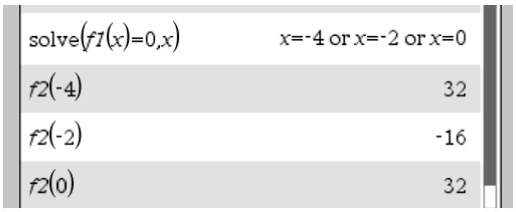
Turning point at $\frac{dy}{dx} = 0$

$x = \frac{11}{3}$

At $x = \frac{11}{3}$, $\frac{d^2y}{dx^2} = +(-) < 0$, so maximum turning point at $(3.67, 0.38)$.

12 TI-Nspire CAS





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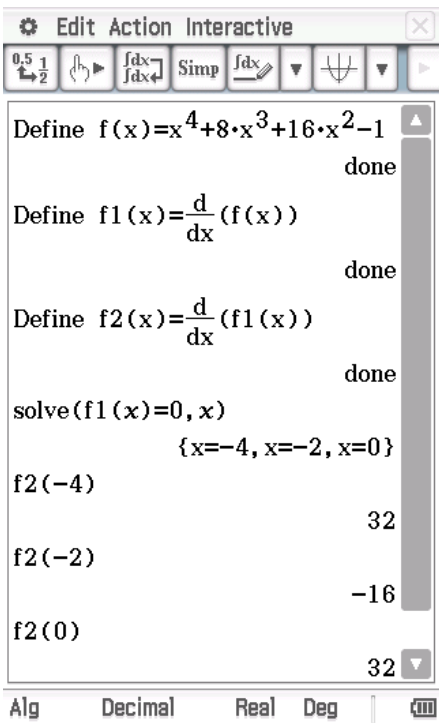

$y = x^4 + 8x^3 + 16x^2 - 1$

$\frac{dy}{dx} = 4x^3 + 24x^2 + 32x$

$\frac{d^2y}{dx^2} = 12x^2 + 48x + 32 = 4(3x^2 + 12x + 8)$

Turning point at $\frac{dy}{dx} = 0$

$4x^3 + 24x^2 + 32x = 0$

$4x\left(x^2 + 6x + 8\right) = 0$

$4x(x+2)(x+4) = 0$

$x = 0$ or $x = -2$ or $x = -4$

At $x = 0$, $\frac{d^2y}{dx^2} = 32 > 0$, so minimum turning point at (0, –1).

At $x = -2$, $\frac{d^2y}{dx^2} = 4(12 - 24 + 8) < 0$ so maximum turning point at (–2, 15).

At $x = -4$, $\frac{d^2y}{dx^2} = 4(8) > 0$, so minimum turning point at (–4, –1).

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Reasoning and communication

13 **a** $y = ax^2 - 4x + 1$

$$\frac{dy}{dx} = 2ax - 4$$

$$\frac{d^2y}{dx^2} = 2a$$

Turning point at $\frac{dy}{dx} = 0$

$2ax - 4 = 0$ at $x = -3$

$-6a = 4$

$$a = -\frac{2}{3}$$

b At $x = -3$, $\frac{d^2y}{dx^2} = 2\left(-\frac{2}{3}\right) < 0$, so maximum turning point.

14 $y = x^3 - mx^2 + 8x - 7$

$$\frac{dy}{dx} = 3x^2 - 2mx + 8$$

$$\frac{d^2y}{dx^2} = 6x - 2m$$

Turning point at $\frac{dy}{dx} = 0$

$3x^2 - 2mx + 8 = 0$ at $x = -1$

$3 + 2m + 8 = 0$

$$m = -\frac{11}{2}$$

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15 $y = ax^3 + bx^2 - x + 5$

$$\frac{dy}{dx} = 3ax^2 + 2bx - 1$$

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Point of inflection at $\frac{d^2y}{dx^2} = 0$ and at $(1, -2)$

$6ax + 2b = 0$ at $x = 1$ $\Rightarrow 6a + 2b = 0$ $\Rightarrow b = -3a$

Substitute $(1, -2)$ in $y = ax^3 + bx^2 - x + 5$

$-2 = a + b - 1 + 5$

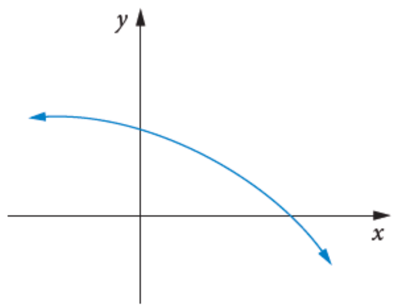
$a + b = -6 \Rightarrow a - 3a = -6$

$a = 3,\ b = -9$

Exercise 3.05 Graph sketching

Concepts and techniques

1 **a**



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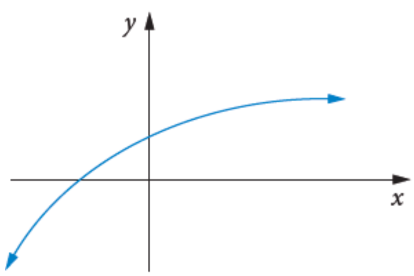

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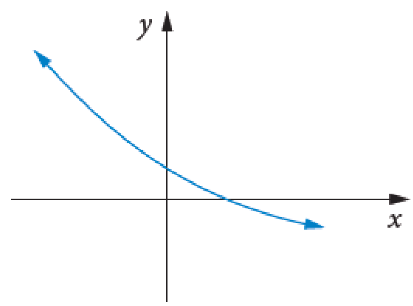

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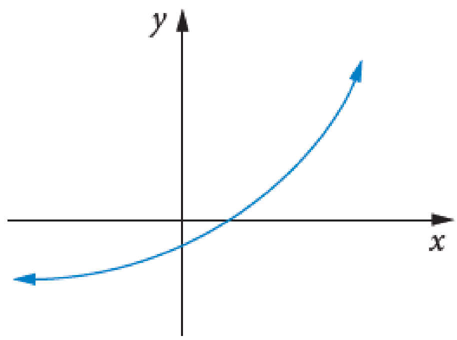

2 **a** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

b $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

c $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

d $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

e $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

3 **a** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

b The rate is decreasing.

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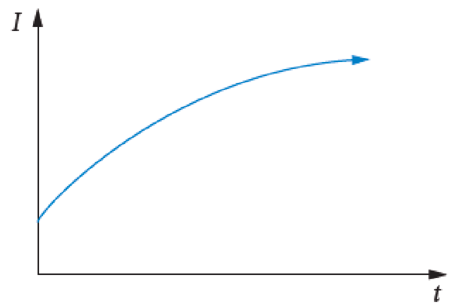
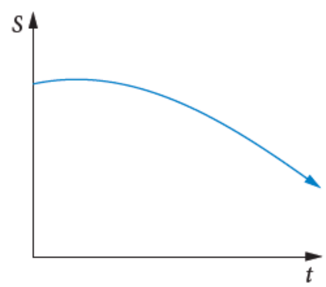
4



Inflation is increasing, but the rate of increase is slowing.

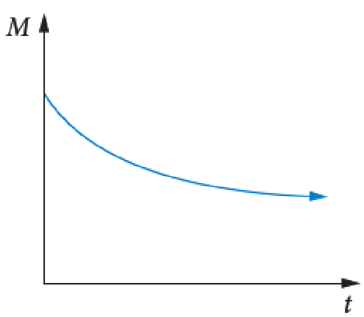
Reasoning and communication

5



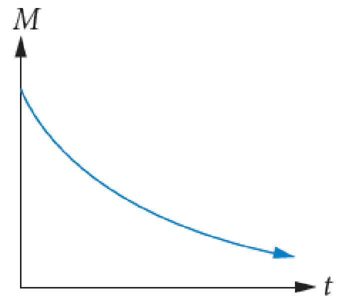
The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing.

6



As an iceblock melts, the rate at which it melts increases.

7 The graph shows the decay of a radioactive substance.

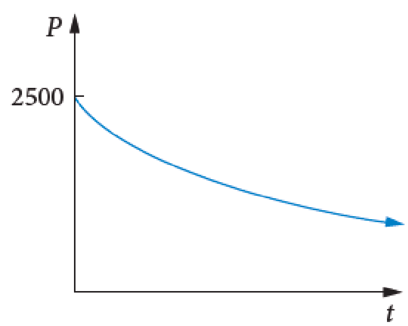


$\frac{dM}{dt} < 0$ and $\frac{d^2M}{dt^2} > 0$. The mass is decreasing but at a decreasing rate.

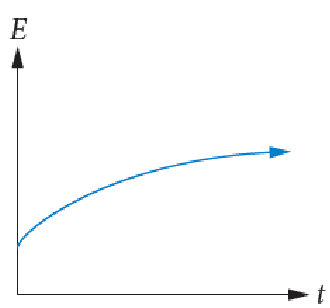
8 **a** The number of fish was decreasing.

b The rate of change of the fish population is increasing.

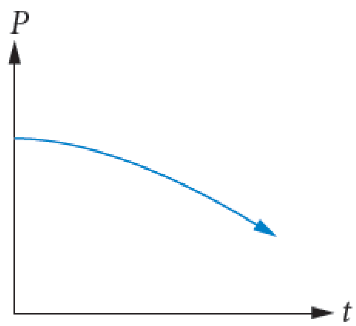
c



9 The level of education of youths in a certain rural area over the past 100 years is increasing at a decreasing rate.



10 The number of students in a high school is decreasing at a decreasing rate.



11 $f(x) = x^2 - 3x - 4$

$f'(x) = 2x - 3$

$f''(x) = 2$

Turning point at $f'(x) = 0$

$x = 1.5$

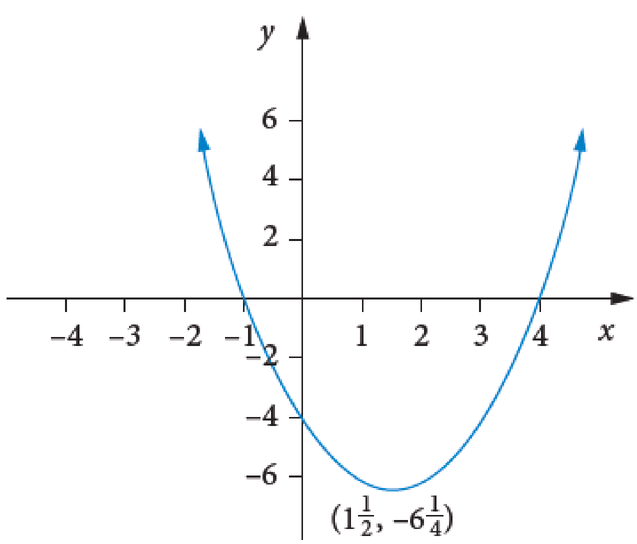
At $x = 1.5$, $f''(1.5) = 2 > 0$ so minimum turning point at $(1.5, -6.25)$.

y intercept: $f(0) = -4$

x intercepts: $0 = x^2 - 3x - 4$

$0 = (x-4)(x+1)$

$(4, 0), (-1, 0)$



12 $y = 6 - 2x - x^2$

$$\frac{dy}{dx} = -2 - 2x$$

$$\frac{d^2y}{dx^2} = -2$$

Turning point at $\frac{dy}{dx} = 0$

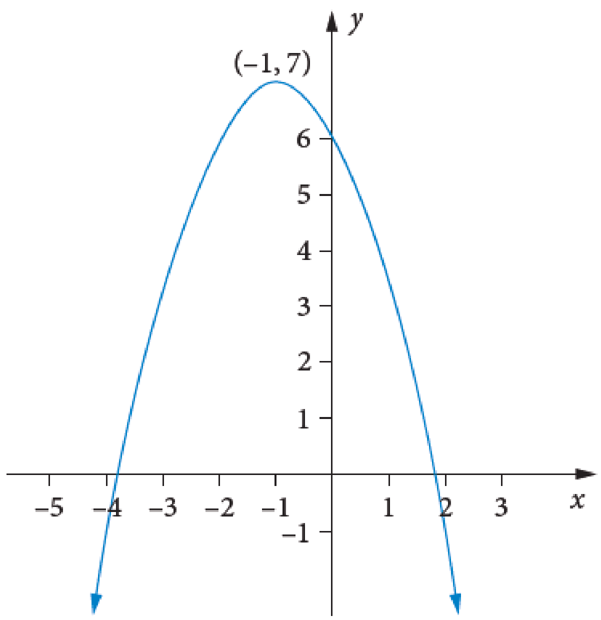
$x = -1$

At $x = -1$, $\frac{d^2y}{dx^2} = -2 < 0$ so maximum turning point at $(-1, 7)$.

y intercept: $f(0) = 6$

x intercepts: $0 = 6 - 2x - x^2$

$(-3.6, 0)$, $(1.6, 0)$



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13 $y = (x - 1)^3$

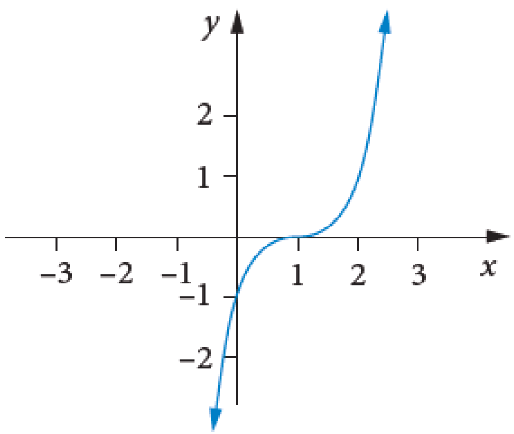
$$\frac{dy}{dx} = 3(x-1)^2$$

$$\frac{d^2y}{dx^2} = 6(x-1)$$

Turning point at $\frac{dy}{dx} = 0$ i.e. at $x = 1$.

At $x = 1$, $\frac{d^2y}{dx^2}$ changes from negative to positive,

the concavity changes so there is a point of inflection at (1, 0).



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14 $y = x^4 + 3$

$$\frac{dy}{dx} = 4x^3$$

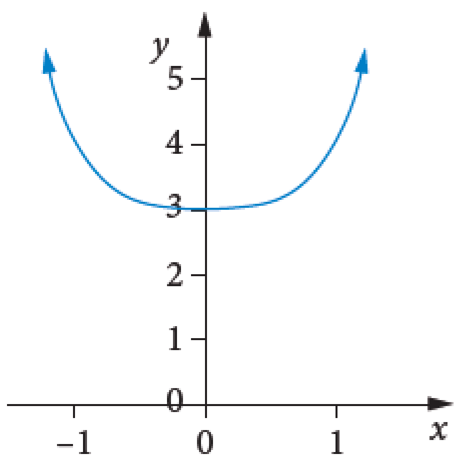
$$\frac{d^2y}{dx^2} = 12x^2$$

Turning point at $\frac{dy}{dx} = 0$ i.e. at $x = 0$.

At $x = 0$, $\frac{d^2y}{dx^2} = 0$ so possibly a stationary point of inflection.

x	–1	0	1
f″(x)	+	0	+

Minimum turning point at (0, 3).



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15 $y = x^5$

$$\frac{dy}{dx} = 5x^4$$

$$\frac{d^2y}{dx^2} = 20x^3$$

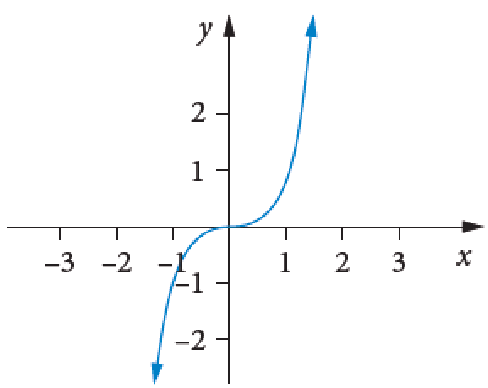
Possible turning point at $\frac{dy}{dx} = 0$ i.e. at $x = 0$.

At $x = 0$, $\frac{d^2y}{dx^2} = 0$ so possibly a stationary point of inflection.

x	–1	0	1
$\frac{d^2y}{dx^2}$	–	0	+

Concave down for $x < 0$ and concave up for $x > 0$.

Stationary point of inflection at (0, 0).



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16 $f(x) = x^7$

$f'(x) = 7x^6$

$f''(x) = 42x^5$

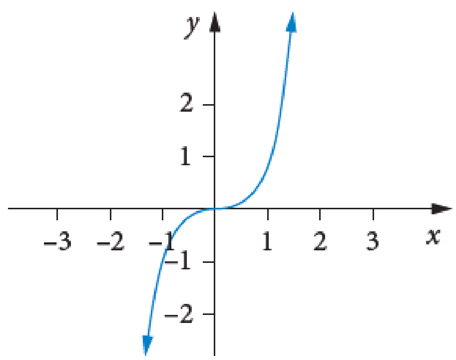
Possible turning point at $f'(x) = 0$ i.e. at $x = 0$.

At $x = 0$, $f''(x) = 0$ so possibly a stationary point of inflection.

x	–1	0	1
$f''(x)$	–	0	+

Concave down for $x < 0$ and concave up for $x > 0$.

Stationary point of inflection at (0, 0).



17 $y = 2x^3 - 9x^2 - 24x + 30$

$$\frac{dy}{dx} = 6x^2 - 18x - 24$$

$$\frac{d^2y}{dx^2} = 12x - 18$$

Turning point at $\frac{dy}{dx} = 0$.

i.e. at $6x^2 - 18x - 24 = 0$

$x^2 - 3x - 4 = 0$

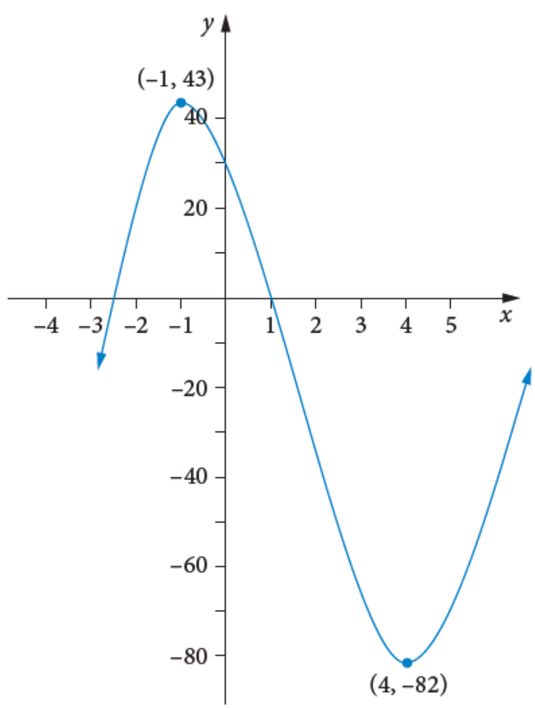
$(x-4)(x+1) = 0$

$x = 4$ or $x = -1$

At $x = 4$, $\frac{d^2y}{dx^2} > 0$ so minimum turning point at $(4, -82)$.

At $x = -1$, $\frac{d^2y}{dx^2} < 0$ so maximum turning point at $(-1, 43)$.

Point of inflection at $\frac{d^2y}{dx^2} = 0$ i.e. at $(1.5, -19.5)$.



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18 **a** $y = x^3 + 6x^2 - 7$

$\frac{dy}{dx} = 3x^2 + 12x$

Turning points at $\frac{dy}{dx} = 0$

i.e. at $3x^2 + 12x = 0$

$3x(x+4) = 0$

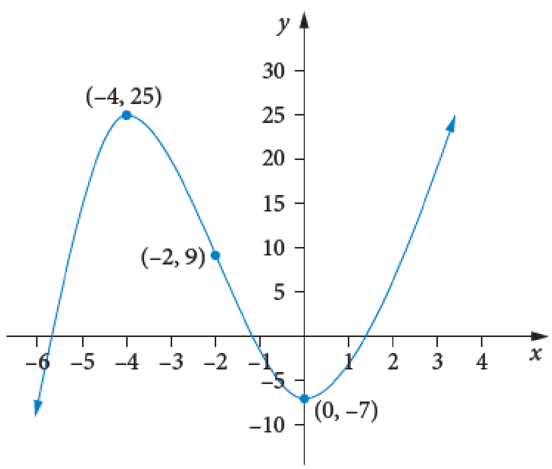
$x = 0$ or $x = -4$

Turning points $(0, -7), (-4, 25)$

b $\frac{d^2y}{dx^2} = 6x + 12$

Point of inflection at $\frac{d^2y}{dx^2} = 0$ i.e. at $(-2, 9)$.

c



19 $y = x^3 - 6x^2 + 3$

$$\frac{dy}{dx} = 3x^2 - 12x$$

Turning points at $\frac{dy}{dx} = 0$

i.e. at $3x^2 - 12x = 0$

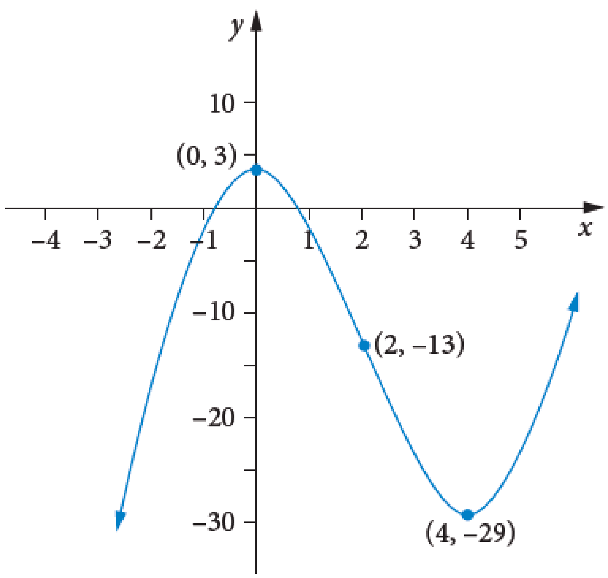
$3x(x-4) = 0$

$x = 0$ or $x = 4$

Turning points $(0, 3), (4, -29)$

$$\frac{d^2y}{dx^2} = 6x - 12$$

Point of inflection at $\frac{d^2y}{dx^2} = 0$ i.e. at $(2, -13)$.



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20 $y = 2 + 9x - 3x^2 - x^3$

$$\frac{dy}{dx} = 9 - 6x - 3x^2$$

$$\frac{d^2y}{dx^2} = -6 - 6x$$

Turning point at $\frac{dy}{dx} = 0$

i.e. at $9 - 6x - 3x^2 = 0$

$-3(x^2 + 2x - 3) = 0$

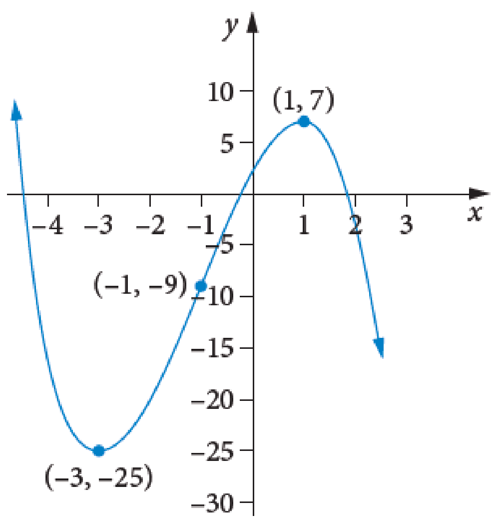
$(x-1)(x+3) = 0$

$x = 1$ or $x = -3$

At $x = 1$, $\frac{d^2y}{dx^2} < 0$ so maximum turning point at $(1, 7)$.

At $x = -3$, $\frac{d^2y}{dx^2} > 0$ so minimum turning point at $(-3, -25)$.

Point of inflection at $\frac{d^2y}{dx^2} = 0$ i.e. at $(-1, -9)$.



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21 $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$

$f'(x) = 12x^3 + 12x^2 - 24x$

$f''(x) = 36x^2 + 24x - 24$

Turning point at $f'(x) = 0$

i.e. at $12x^3 + 12x^2 - 24x = 0$

$12x(x^2 + x - 2) = 0$

$x(x-1)(x+2) = 0$

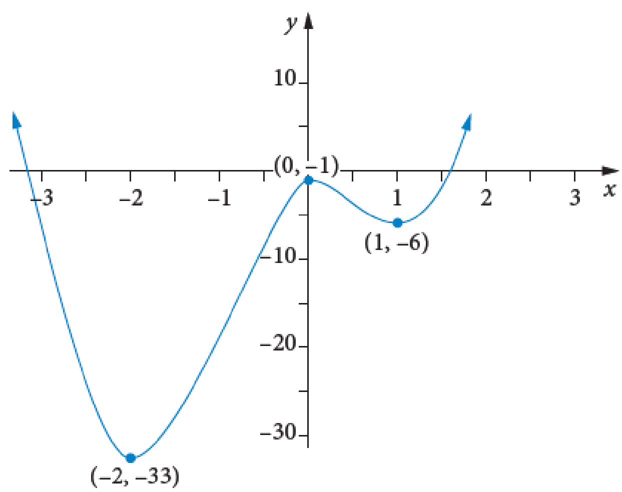
$x = 0$ or $x = 1$ or $x = -2$

At $x = 0$, $f''(x) < 0$ so maximum turning point at $(0, -1)$.

At $x = 1$, $f''(x) > 0$ so minimum turning point at $(1, -6)$.

At $x = -2$, $f''(x) > 0$ so minimum turning point at $(-2, -33)$.

Point of inflection at $f''(x) = 0$ i.e. at $(-1, -14)$.



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22 $y = (x - 4)(x + 2)^2$

$$\frac{dy}{dx} = 1(x+2)^2 + 2(x+2)(x-4)$$
$$= (x+2)(x+2+2x-8)$$
$$= (x+2)(3x-6)$$

$$\frac{d^2y}{dx^2} = 1(3x-6) + 3(x+2)$$
$$= 6x$$

Turning point at $\frac{dy}{dx} = 0$

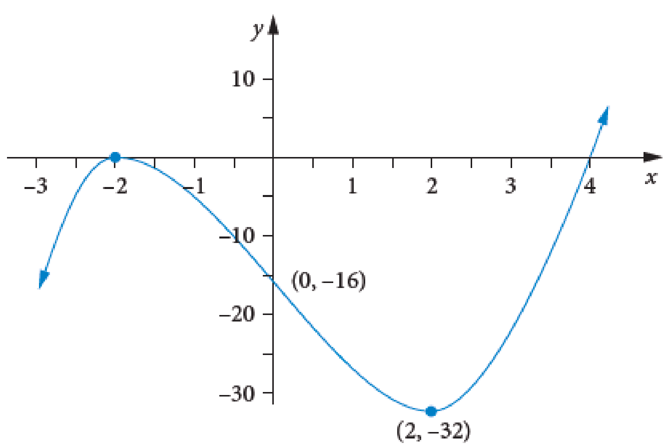
i.e. at $(x+2)(3x-6) = 0$

$x = -2$ or $x = 2$

At $x = -2$, $\frac{d^2y}{dx^2} < 0$ so maximum turning point at $(-2, 0)$.

At $x = 2$, $\frac{d^2y}{dx^2} > 0$ so minimum turning point at $(2, -32)$.

Point of inflection at $\frac{d^2y}{dx^2} = 0$ i.e. at $(0, -16)$.



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23 $y = (2x + 1)(x - 2)^4$

$$\frac{dy}{dx} = 2\left(x-2\right)^4 + 4(x-2)^3\left(2x+1\right)$$
$$= (x-2)^3(2x-4+8x+4)$$
$$= (x-2)^3\left(10x\right)$$
$$= 10x(x-2)^3$$
$$\frac{d^2y}{dx^2} = 10(x-2)^3 + 3\left(x-2\right)^2 10x$$
$$= 10(x-2)^2\left(x-2+3x\right)$$
$$= 10(x-2)^2\left(4x-2\right)$$
$$= 20(x-2)^2\left(2x-1\right)$$

Turning point at $\frac{dy}{dx} = 0$

i.e. at $20(x-2)^2\left(2x-1\right) = 0$

$x = 2$ or $x = \frac{1}{2}$

Turning points at (2, 0) and (0.5, 10.125).

At $x = 2$, $\frac{d^2y}{dx^2} = 0$ so possible stationary point of inflection at (2, 0).

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 0$ so possible stationary point of inflection at (0.5,10.125).

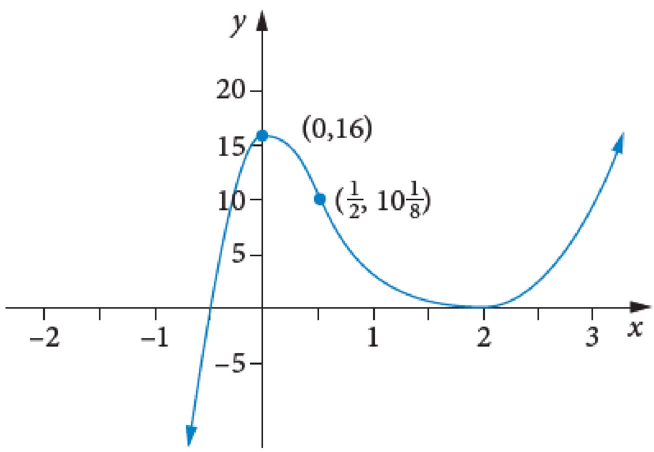
x	1	2	3
$\frac{d^2y}{dx^2}$	+	0	+

Concave up for $x < 2$ and concave up for $x > 2$. Turning point at (2, 0).

x	0	0.5	1
$\frac{d^2y}{dx^2}$	–	0	+

Concave down for $x < 0.5$ and concave up for $x > 0.5$. Point of inflection at (0.5, 0.125).

y-intercept (0, 16)



Exercise 3.06 Optimisation

Concepts and techniques

1 Let a and b be the two numbers.

$ab = 27$

Let $S = 3a + 4b$

$\therefore \; S = 3a + 4\left(\frac{27}{a}\right)$

$S = 3a + \frac{108}{a}$

Minimum S when $\frac{dS}{da} = 0$ and $\frac{d^2S}{da^2} > 0$

$\frac{dS}{da} = 3 - \frac{108}{a^2}$

$\frac{d^2S}{da^2} = 2 \times \frac{108}{d^3}$

$\frac{d^2S}{da^2} = \frac{216}{d^3}$

At $\frac{dS}{da} = 0$, $3 - \frac{108}{a^2} = 0$

$a^2 = 36$

$a > 0$ so $a = 6$

Check for minimum

$\frac{d^2S}{da^2} = \frac{216}{6^3} > 0$ so minimum

$a = 6$, $b = ?$

$ab = 27 \Rightarrow b = 4.5$

The two numbers are 6 and 4.5.

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2 Let x and y be the two numbers. $x + y = 25$

Let P be the product. $P = xy$. Want maximum P.

$P = x(25 - x)$

Maximum P when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$\frac{dP}{dx} = (25 - x) + (-1)x = 25 - 2x$

$\frac{d^2P}{dx^2} = -2$

At $\frac{dP}{dx} = 0$, $x = 12.5$

$\frac{d^2P}{dx^2} = -2 < 0$ so maximum

$x = 12.5$, $y = ?$

$x + y = 25 \Rightarrow y = 12.5$

The two numbers are 12.5 and 12.5.

3 Let x and y be the two numbers. $x - y = 40$

Let P be the product. $P = xy$. Want minimum P.

$P = x(x - 40) = x^2 - 40x$

Minimum P when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$\frac{dP}{dx} = 2x - 40$

$\frac{d^2P}{dx^2} = 2$

At $\frac{dP}{dx} = 0$, $x = 20$

$\frac{d^2P}{dx^2} = 2 > 0$ so minimum

$x = 20$, $y = ?$

$x - y = 40 \Rightarrow y = -20$

The two numbers are 20 and –20.

4 Let x and y be the two numbers. $x + y = 32$

Let S be the sum of the squares. $S = x^2 + y^2$. Want minimum S.

$S = x^2 + (32 - x)^2 = 2x^2 - 64x + 1024$

Minimum S when $\frac{dS}{dx} = 0$ and $\frac{d^2S}{dx^2} < 0$

$\frac{dS}{dx} = 4x - 64$

$\frac{d^2S}{dx^2} = 4$

At $\frac{dS}{dx} = 0$, $x = 16$

$\frac{d^2S}{dx^2} = 4 > 0$ so minimum at $x = 16$

$x = 16$, $y = ?$

$x + y = 32 \Rightarrow y = 16$

The two numbers are both 16.

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5 Let the distance of the point from the origin be *d*.

$$d = \sqrt{x^2 + y^2} \quad \text{but } y = 7 - x$$
$$= \sqrt{x^2 + (7 - x)^2}$$
$$= \sqrt{2x^2 - 14x + 49}$$

Minimum d when $\frac{dd}{dx} = 0$ and $\frac{d^2d}{dx^2} < 0$

$$\frac{dd}{dx} = \frac{1}{2}\left(2x^2 - 14x + 49\right)^{-\frac{1}{2}}\left(4x - 14\right)$$
$$= \frac{2x - 7}{\sqrt{2x^2 - 14x + 49}}$$

$$\frac{d^2d}{dx^2} = \frac{2\sqrt{2x^2 - 14x + 49} - \frac{1}{2}\left(2x^2 - 14x + 49\right)^{-\frac{1}{2}}\left(2x - 7\right)}{2x^2 - 14x + 49}$$
$$= \frac{1}{2x^2 - 14x + 49} \times \left[2\sqrt{2x^2 - 14x + 49} - \frac{1}{2}\left(2x^2 - 14x + 49\right)^{-\frac{1}{2}}\left(2x - 7\right)\right]$$
$$= \frac{1}{2x^2 - 14x + 49} \times \left[\frac{4\left(2x^2 - 14x + 49\right) - (2x - 7)}{2\sqrt{2x^2 - 14x + 49}}\right]$$
$$= \frac{1}{2x^2 - 14x + 49} \times \left[\frac{8x^2 - 58x + 203)}{2\sqrt{2x^2 - 14x + 49}}\right]$$

At $\frac{dd}{dx} = 0$, $x = 3.5$

At $x = 3.5$, $\frac{d^2d}{dx^2} > 0$ so minimum

$x = 3.5$, $y = ?$

$y = 7 - x \Rightarrow y = 3.5$

$P(3.5, 3.5)$

6 $d^2 = (x - 4)^2 + (y - 0)^2$ but $y = \sqrt{x}$

$d^2 = x^2 - 8x + 16 + x$

$d^2 = x^2 - 7x + 16$

$d = \sqrt{x^2 - 7x + 16}$

Minimum d when $\frac{dd}{dx} = 0$ and $\frac{d^2d}{dx^2} < 0$

$$\frac{dd}{dx} = \frac{1}{2}\left(x^2 - 7x + 16\right)^{-\frac{1}{2}}\left(2x - 7\right)$$

$$= \frac{2x - 7}{\sqrt{x^2 - 7x + 16}}$$

$$\frac{d^2d}{dx^2} = \frac{2\sqrt{x^2 - 7x + 16} - \frac{1}{2}\left(x^2 - 7x + 16\right)^{-\frac{1}{2}}\left(2x - 7\right)\left(2x - 7\right)}{x^2 - 7x + 16}$$

$$= \frac{1}{x^2 - 7x + 16} \times \left[2\sqrt{x^2 - 7x + 16} - \frac{1}{2}\left(x^2 - 7x + 16\right)^{-\frac{1}{2}}\left(2x - 7\right)^2\right]$$

$$= \frac{1}{x^2 - 7x + 16} \times \left[\frac{4\left(x^2 - 7x + 16\right) - (2x - 7)^2}{2\sqrt{x^2 - 7x + 16}}\right]$$

$$= \frac{1}{x^2 - 7x + 16} \times \left[\frac{4x^2 - 28x + 64 - \left(4x^2 - 28x + 49\right)}{2\sqrt{x^2 - 7x + 16}}\right]$$

$$= \frac{1}{x^2 - 7x + 16} \times \left[\frac{15}{2\sqrt{x^2 - 7x + 16}}\right]$$

At $\frac{dd}{dx} = 0$, $x = 3.5$

At $x = 3.5$, $\frac{d^2d}{dx^2} > 0$ so minimum

$x = 3.5$, $y = ?$

$y = \sqrt{x} \Rightarrow y = \sqrt{3.5}$

$(3.5, \sqrt{3.5})$

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7 $f\left(x\right)=\dfrac{6}{x^2+3}$

Slope, *s*, is *f'*(*x*)

$$s(x)=-6(x^2+3)^{-2}2x=-12x(x^2+3)^{-2}$$

Maximum s when $s'(x)=0$ and $s''(x)<0$

Minimum s when $s'(x)=0$ and $s''(x)>0$

$$\begin{aligned}s'(x)&=-12\left[1\times(x^2+3)^{-2}+(-2)(x^2+3)^{-3}(2x)x\right]\\&=-12\left[(x^2+3)^{-3}\right]\left[\left(x^2+3\right)^1-4x^2\right]\\&=-12\left[(x^2+3)^{-3}\right]\left(3-3x^2\right)\\&=36\left[(x^2+3)^{-3}\right]\left(x^2-1\right)\end{aligned}$$

$$s'(x)=\frac{36\left(x^2-1\right)}{(x^2+3)^3}$$

$$\begin{aligned}s''(x)&=36\left[\frac{2x(x^2+3)^3-3(x^2+3)^2 2x\left(x^2-1\right)}{(x^2+3)^6}\right]\\&=\frac{72x(x^2+3)^2}{(x^2+3)^6}\left[x^2+3-3\left(x^2-1\right)\right]\\&=\frac{72x}{(x^2+3)^4}\left(-2x^2+6\right)\\&=\frac{-144x(x^2-3)}{(x^2+3)^4}\end{aligned}$$

$s'(x)=0$ when $\dfrac{36\left(x^2-1\right)}{(x^2+3)^3}$ i.e. at $x=\pm1$

At $x=-1$, $s''(-1)<0$ so maximum gradient $s(-1)=\dfrac{3}{4}$ at $(-1, 1.5)$.

At $x=1$, $s''(1)>0$ so minimum gradient $s(1)=-\dfrac{3}{4}$ at $(1, 1.5)$.

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a The maximum slope is $\frac{3}{4}$ at (–1, 1.5).

$$y = \frac{3x}{4} + c$$

At (–1, 1.5), $\frac{3}{2} = -\frac{3}{4} + c$

$$c = \frac{9}{4}$$

$$y = \frac{3}{4}x + \frac{9}{4}$$

$$4y = 3x + 9$$

b The minimum slope is $-\frac{3}{4}$ at (1, 1.5).

$$y = -\frac{3x}{4} + c$$

At (1, 1.5), $\frac{3}{2} = -\frac{3}{4} + c$

$$c = \frac{9}{4}$$

$$y = -\frac{3}{4}x + \frac{9}{4}$$

$$4y = -3x + 9$$

8 Total weight $W(n) = n \times w(n) = n(600 - 30n)$. Maximum $W(n)$ = ?

$W(n) = 600n - 30n^2$

Maximum $W(n)$ when $W'(n) = 0$ and $W''(n) < 0$.

$W'(n) = 600 - 60n$
$W''(n) = -60$
When $W'(n) = 0$, $n = 10$
$W''(10) < 0$ so maximum.

n =10 leads to the maximum total production of weight in the fish.

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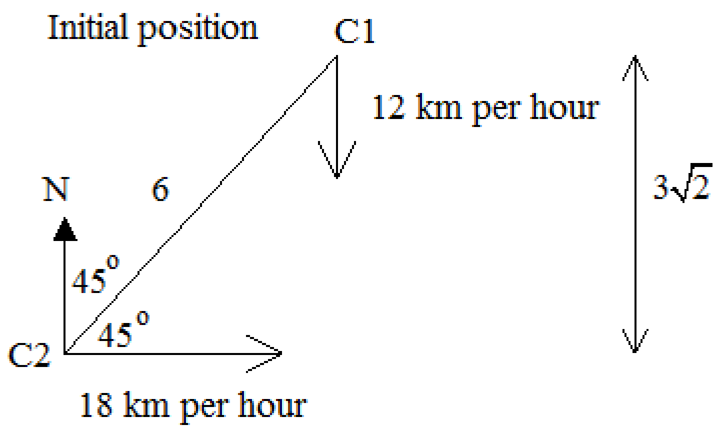
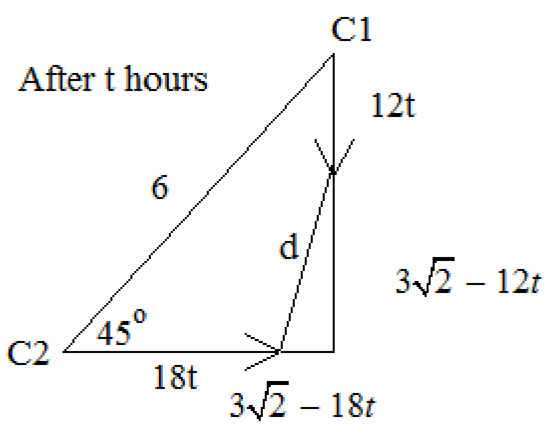
9





Initially the two catamarans are 6 kms apart, making an isosceles triangle with side $3\sqrt{2}$ north–south.

$d^2 = (3\sqrt{2} - 12t)^2 + (3\sqrt{2} - 18t)^2$

$= 18 - 72\sqrt{2}\,t + 144t^2 + 18 - 108\sqrt{2}\,t + 324t^2$

$= 36 - 180\sqrt{2}\,t + 468t^2$

$d = \sqrt{36 - 180\sqrt{2}t + 468t^2}$

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Minimum distance apart when $\frac{dd}{dt}=0$ and $\frac{d^2d}{dt^2}>0$

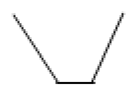
$$\frac{dd}{dt}=\frac{1}{2}\left(36 - 180\sqrt{2}t + 468t^2\right)^{-\frac{1}{2}}\times\left(-180\sqrt{2}+936t\right)$$

$$=\frac{-90\sqrt{2}+468t}{\sqrt{\left(36 - 180\sqrt{2}t + 468t^2\right)}}$$

At $\frac{dd}{dt}=0,\ -90\sqrt{2}+468t=0 \Rightarrow t=0.27 \quad \text{(only)}$

Using the sign test

t	0	0.27	1
$f'(t)$	–	0	+



Therefore minimum at $t\approx 2.7$

$$d=\sqrt{36 - 180\sqrt{2}t + 468t^2}$$

Minimum distance apart is 1.18 km.

10 $q = 8 + \frac{v^2}{50}$

Time for trip $= \frac{d}{v}$

Fuel used $F \; = q \; \frac{d}{v}$

$$= \left(8 + \frac{v^2}{50}\right)\frac{d}{v}$$

$$= \frac{8d}{v} + \frac{vd}{50}$$

v = ? for minimum F

$F = \frac{8d}{v} + \frac{vd}{50}$ where d is a constant (the distance travelled)

$\frac{dF}{dv} = -\frac{8d}{v^2} + \frac{d}{50}$

$\frac{d^2F}{dv^2} = \frac{8d}{v^3} > 0 \quad \text{for } v > 0$

Minimum v when $\frac{dF}{dv} = 0$ and $\frac{d^2F}{dv^2} > 0$

At $\frac{dF}{dv} = 0, \frac{8d}{v^2} = \frac{d}{50} \;\; \Rightarrow \;\; v^2 = 400$

$v = 20$

The speed of the boat for which the amount of fuel used for any given journey is least is 20 km/hr.

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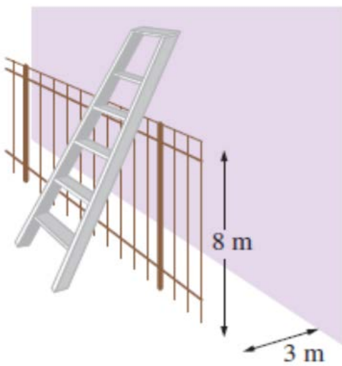
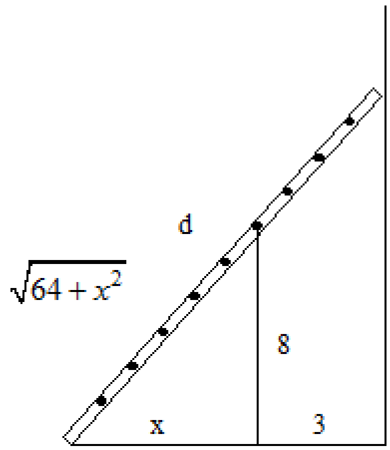
11 Using similar triangles, we have $\dfrac{d}{\sqrt{64+x^2}}=\dfrac{x+3}{x} \Rightarrow d=\dfrac{(x+3)\sqrt{64+x^2}}{x}$





Minimum d when $\dfrac{dd}{dx}=0$ **and** $\dfrac{d^2d}{dx^2}>0$

$$d=\frac{(x+3)\sqrt{64+x^2}}{x}=(1+3x^{-1})\sqrt{64+x^2}$$

$$\frac{dd}{dx}=-\frac{3}{x^2}\sqrt{64+x^2}+\frac{1}{2}\left(64+x^2\right)^{-\frac{1}{2}}\left(2x\right)(1+3x^{-1})$$

$$=-\frac{3}{x^2}\sqrt{64+x^2}+\frac{1}{\sqrt{64+x^2}}\left(x\right)(1+3x^{-1})$$

$$=\frac{3}{x^2}\sqrt{64+x^2}+\frac{x+3}{\sqrt{64+x^2}}$$

$$=\left[\frac{-3\left(64+x^2\right)+x^2(x+3)}{x^2\sqrt{64+x^2}}\right]$$

$$=\left(\frac{x^3-192}{x^2\sqrt{64+x^2}}\right)$$

If $\dfrac{dd}{dx}=0,\ x=5.77$

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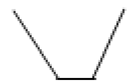
Use the sign test

x	0	5.77	6
$f'(x)$	–	0	+



Therefore minimum at $x \approx 5.77$, d = ?

$$d = \frac{(x+3)\sqrt{64+x^2}}{x} \approx 14.99$$

The shortest ladder is about 15 metres.

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12 Let $P_{\text{hour}} = 250\ 000 + v^3$

$P_{\text{distance between stations}} = (250\ 000 + v^3) \times t$

$$P_{\text{d}} = (250\ 000 + v^3) \times \frac{d}{v}$$

$$P_{\text{d}} = \left(\frac{250\ 000}{v} + v^2\right)d$$

To minimise power P, we need $\frac{dP}{dv} = 0$ and $\frac{dP}{dv} > 0$ and $v > 0$.

$$\frac{dP_d}{dv} = \left(-\frac{250\ 000}{v^2} + 2v\right)d$$

$$\frac{d^2P_d}{dv^2} = \left(\frac{500\ 000}{v^3} + 2\right)d$$

If $\frac{dP_d}{dv} = 0$, then $\frac{250\ 000}{v^2} = 2v \Rightarrow v = 50$

The speed at which the train should travel to minimise the use of electricity

between stations is 50 km/hr.

13 $y = 1.4 + x - 0.04x^2$

$$\frac{dy}{dx} = 1 - 0.008x$$

$\frac{d^2y}{dx^2} = -0.008 < 0$ so any turning point will be a maximum.

Turning point at $\frac{dy}{dx} = 0$

i.e. at $x = 125$

$y = 63.9$

The greatest height reached by the ball is 63.9 m.

14 **a** $p = \dfrac{t^2}{5\left(1+t^2\right)^2}$

$$\frac{dp}{dt} = \frac{1}{5}\left[\frac{2t\left(1+t^2\right)^2 - 2\left(1+t^2\right)2t \times t^2}{\left(1+t^2\right)^4}\right]$$

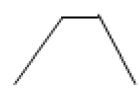
$$= \frac{2t\left(1+t^2\right)\left(1+t^2-2t^2\right)}{5\left(1+t^2\right)^4}$$

$$= \frac{2t\left(1-t^2\right)}{5\left(1+t^2\right)^3}$$

If $\dfrac{dp}{dt} = 0,\ t = 0, \pm 1$

Test t for $t > 0$

x	0.5	1	2
$\boldsymbol{\dfrac{dp}{dt}}$	+	0	–



Maximum at $x = 1$, i.e. $p = \dfrac{1}{20}$

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b t = ? when increasing most rapidly.

Want maximum $\frac{dp}{dt}$

$$\frac{dp}{dt}=\frac{2t\left(1-t^2\right)}{5\left(1+t^2\right)^3}=r$$

Maximum r when $\frac{dr}{dt}=0$

$$\frac{dr}{dt}=\frac{2}{5}\times\left\{\frac{\left[1\times\left(1-t^2\right)+(-2t)t\right]\left(1+t^2\right)^3-3\left(1+t^2\right)^2 2t\times t\left(1-t^2\right)}{\left(1+t^2\right)^6}\right\}$$

$$=\frac{2\left(1+t^2\right)^2}{5\left(1+t^2\right)^6}\times\left\{\left[\left(1-t^2\right)-2t^2\right]\left(1+t^2\right)-6t^2\left(1-t^2\right)\right\}$$

$$=\frac{2}{5\left(1+t^2\right)^4}\times\left[\left(1-3t^2\right)\left(1+t^2\right)-6t^2+6t^4\right]$$

$$=\frac{2}{5\left(1+t^2\right)^4}\times\left(1-8t^2+3t^4\right)$$

$$=\frac{2\left(1-8t^2+3t^4\right)}{5\left(1+t^2\right)^4}$$

If $\frac{dr}{dt}=0,\ t^2=0.131\ 482\ 9082$ or $t^2=2.535$

but $0 < t < 1$ and $t > 0$, $t=\sqrt{0.131\ 482\ 9082}=0.363$ months (or 10 or 11 days)

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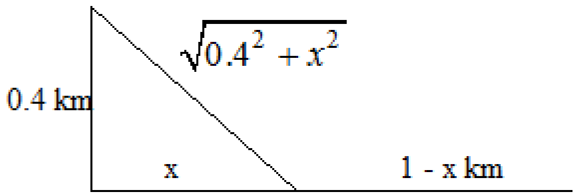
15



$$v = \frac{x}{t}$$

$$T = T_{\text{swimming}} + T_{\text{running}}$$

$$\text{time} = \frac{\text{distance}}{\text{velocity}}$$

$$T = \frac{\sqrt{0.4^2 + x^2}}{4} + \frac{1-x}{12}$$

Minimum time T when $\frac{dT}{dx} = 0$ and $\frac{d^2T}{dx^2} > 0$

$$\frac{dT}{dx} = \frac{1}{4}\left[\frac{1}{2}\left(0.4^2 + x^2\right)^{-\frac{1}{2}}(2x)\right] - \frac{1}{12}$$

$$= \frac{1}{4}\left(\frac{2x}{2\sqrt{0.4^2 + x^2}}\right) - \frac{1}{12}$$

When $\frac{dT}{dx} = 0, \frac{x}{4\sqrt{0.4^2 + x^2}} = \frac{1}{12}$

$$3x = \sqrt{0.4^2 + x^2}$$

$$0.4^2 + x^2 = 9x^2$$

$$8x^2 = 0.16$$

$$x > 0$$

$$x = 0.141\ 42$$

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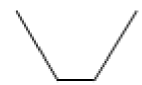
x	0	0.141 42	1
$\frac{dp}{dt}$	–	0	+



Concave up so minimum at $x = 0.14142$

$\tan(\theta) = \frac{0.14142}{0.4} \Rightarrow \theta = 19.5° \Rightarrow$ The angle he should make with the beach is 70.5°.

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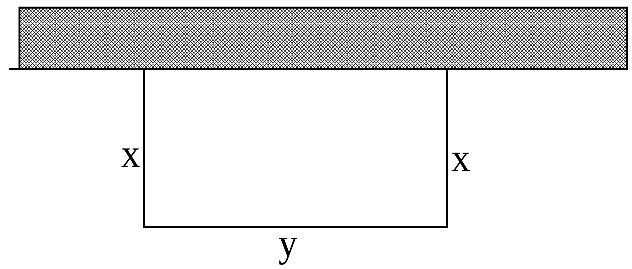
Exercise 3.07 Optimisation in area and volume

Reasoning and communication

1



$2x + y = 2000$

$A = xy = x(2000 - 2x)$

$= 2000x - 2x^2$

Maximum A when $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$

$\frac{dA}{dx} = 2000 - 4x$

$\frac{d^2y}{dx^2} = -4$

At $\frac{dy}{dx} = 0$, $x = 500$

At $x = 500$, $\frac{d^2y}{dx^2} < 0$ so maximum area at (500, 1000).

The maximum possible area is 500 000 m^2 and dimensions of the paddock

are 500 m × 1000 m.

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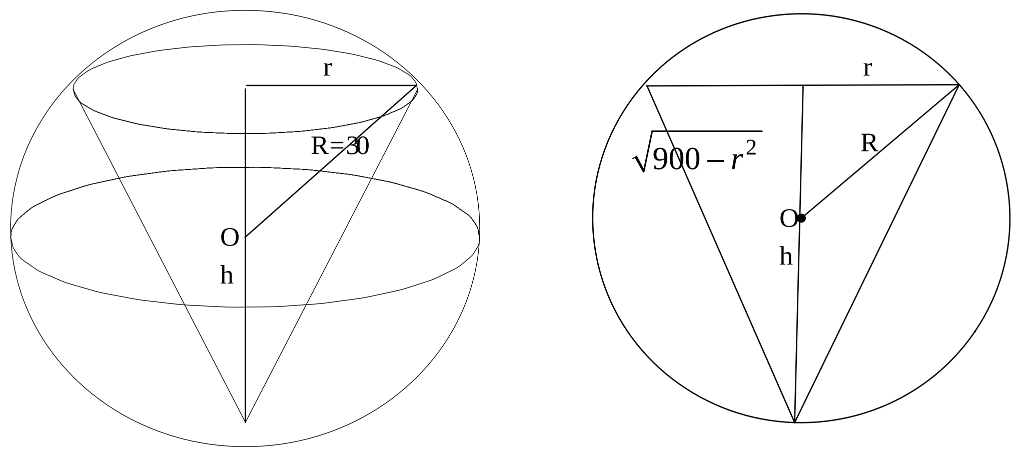
2



$h = 30 + \sqrt{900 - r^2}$

$(h - 30)^2 = 900 - r^2$

$r^2 = 900 - (h - 30)^2$

$r^2 = 900 - h^2 + 60h - 900$

$r^2 = -h^2 + 60h$

$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(-h^2 + 60h)h$

$V_{\text{cone}} = \frac{1}{3}\pi (60h^2 - h^3)$

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Maximum volume of cone when $\frac{dV}{dh}=0$ and $\frac{d^2V}{dh^2}<0$

$$\frac{dV}{dh}=\frac{\pi}{3}\left(120h-3h^2\right)$$

$$\frac{d^2V}{dh^2}=\frac{\pi}{3}\left(120-6h\right)$$

At $\frac{dV}{dh}=0$, $\quad 120h=3h^2$

$h=0,40 \qquad\qquad h\neq 0$

$h=40, \quad \frac{d^2V}{dh^2}<0$ so maximum volume

At h = 40, $r^2 = -h^2 + 60h \Rightarrow r = \sqrt{800}$

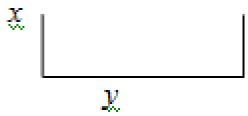
$$\frac{V_{\text{cone}}}{V_{\text{sphere}}}=\frac{\frac{1}{3}\pi r^2 h}{\frac{4}{3}\pi r^3}$$

$$=\frac{h}{4r}$$

$$=\frac{40}{4\times\sqrt{800}}$$

$$=1:2\sqrt{2}\approx 1:2.83$$

3



$2x + y = 24 \Rightarrow y = 24 - 2x$

Maximum volume when cross-section of the end is maximised.

$A = xy = x(24 - 2x) = 24x - 2x^2$

Maximum A when $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$

$\frac{dA}{dx} = 24 - 4x$

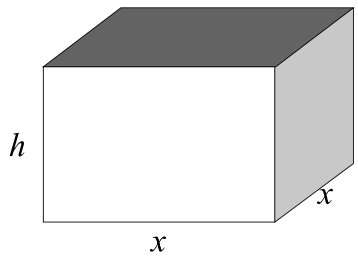
$\frac{d^2A}{dx^2} = -4$

At $\frac{dA}{dx} = 0$, $x = 6$

At $x = 6$, $\frac{d^2A}{dx^2} < 0$ so maximum area

The dimensions of the cross-section of the guttering if it is to hold the maximum volume of water are 6×12 cm ($d \times w$).

4



$500 = x^2h$

$M = x^2 + 4hx$, but $h = \dfrac{500}{x^2}$

$$M = x^2 + \frac{2000}{x}$$

Minimum M when $\dfrac{dM}{dx} = 0$ and $\dfrac{d^2M}{dx^2} > 0$

$\dfrac{dM}{dx} = 2x - \dfrac{2000}{x^2}$

$\dfrac{d^2M}{dx^2} = 2 + \dfrac{4000}{x^3} > 0$ for $x > 0$

At $\dfrac{dM}{dx} = 0$, $x^3 = 1000 \Rightarrow x = 10$

At $x = 10$, $\dfrac{d^2M}{dx^2} > 0$ so minimum M

$500 = x^2h \Rightarrow h = 5$

For the least amount of material to be used, the square base must be 10 cm × 10 cm

and the height must be 5 cm.

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5

$36 = xh^2$

Let $M = 2h^2 + 3hx$, but $x = \dfrac{36}{h^2}$

$M = 2h^2 + \dfrac{108}{h}$

Minimum M when $\dfrac{dM}{dx} = 0$ and $\dfrac{d^2M}{dx^2} > 0$

$\dfrac{dM}{dx} = 4h - \dfrac{108}{h^2}$

$\dfrac{d^2M}{dx^2} = 4 + \dfrac{216}{h^3} > 0$ for $h > 0$

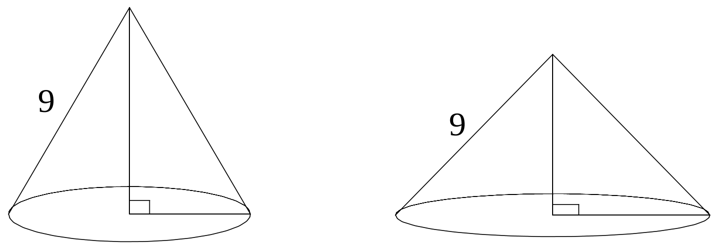
At $\dfrac{dM}{dx} = 0$, $4h^3 = 108 \Rightarrow h = 3$

At $h = 3$, $\dfrac{d^2M}{dx^2} > 0$ so minimum M

$M = 2h^2 + \dfrac{108}{h} = 2 \times (3)^2 + \dfrac{108}{3} = 54$

Minimum M is 54 m^2.

6



$$V = \frac{1}{3}\pi r^2 h \quad \text{and} \quad 9^2 = r^2 + h^2$$
$$= \frac{1}{3}\pi(81 - h^2)h$$
$$= 27\pi h - \frac{h^3\pi}{3}$$

Maximum V when $\frac{dV}{dr} = 0$ and $\frac{d^2V}{dr^2} > 0$

$$\frac{dV}{dr} = 27\pi - h^2\pi$$

$$\frac{d^2V}{dr^2} = -2h < 0 \text{ for } h > 0$$

At $\frac{dV}{dr} = 0$, $h^2 = 27 \Rightarrow h = 5.196$

At $h = 5.196$, $\frac{d^2V}{dr^2} < 0$ so maximum V

The height of the cone that will have the greatest volume is 5.2 cm.

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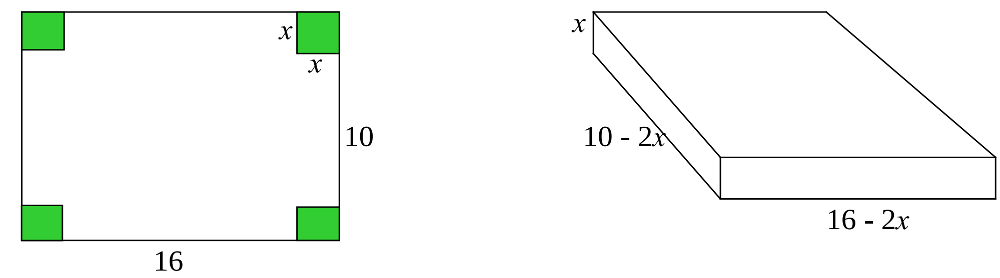
7



$V_{box} = x(10 - 2x)(16 - 2x)$, x = ? for max V

$V = 4[x(5 - x)(8 - x)] = 4(40x - 13x^2 + x^3)$

Maximum volume when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0$

$\frac{dV}{dx} = 160 - 104x + 12x^2$

$\frac{d^2V}{dx^2} = -104 + 24x$

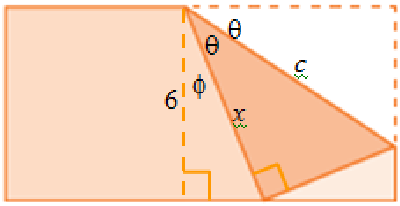
At $\frac{dV}{dx} = 0$, $160 - 104x + 12x^2 = 0$

$3x^2 - 26x + 40 = 0 \Rightarrow x = 2$ or $x = \frac{20}{3}$ but because one side is 10 cm, $0 < x < 5$

At $h = 2$, $\frac{d^2V}{dx^2} < 0$ so maximum volume

Size of square is 2 cm $\times$ 2 cm

8



Let the angle of turndown be θ and the remaining part ϕ as shown on the diagram.

Clearly $0° < \theta \le 45°$

Then $\phi + 2\theta = 90°$, so $\phi = 90° - 2\theta$

Let c be the length of the crease and x be the length of paper turned from the top.

Then $c = \dfrac{x}{\cos(\theta)} = \dfrac{6}{\cos(\theta)\cos(\phi)} = \dfrac{6}{\cos(\theta)\cos(90° - 2\theta)} = \dfrac{6}{\cos(\theta)\sin(2\theta)}$

Now $\dfrac{dc}{d\theta} = \dfrac{0 - 6[-\sin(\theta)\times\sin(2\theta) + \cos(\theta)\times 2\cos(2\theta)]}{[\cos(\theta)\sin(2\theta)]^2}$

$= \dfrac{-6[\cos(\theta)\cos(2\theta) + \cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta)]}{\cos^2(\theta)\sin^2(2\theta)}$

$= \dfrac{-6[\cos(\theta)\cos(2\theta) + \cos(3\theta)]}{\cos^2(\theta)\sin^2(2\theta)}$

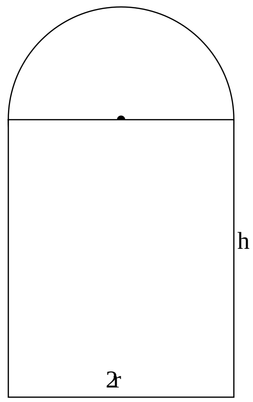
$\dfrac{dc}{d\theta} = 0$ when $\cos(\theta)\cos(2\theta) + \cos(3\theta) = 0$

Solving on a CAS calculator in the domain $0° < \theta \le 45°$ gives $\theta = 35.264\ldots°$

$\left.\dfrac{dc}{d\theta}\right|_{\theta=35°} = -0.0037\ldots < 0$ and $\left.\dfrac{dc}{d\theta}\right|_{\theta=36°} = 0.0104\ldots > 0$ so there is a minimum at $35.264\ldots°$

$c(35.264\ldots°) = 7.794\ldots$, so the minimum length of the crease is about 7.79 cm.

9



Perimeter = 12 m

$P = 0.5(2\pi r) + 2h + 2r$

$12 = \pi r + 2h + 2r \Rightarrow h = 0.5(12 - \pi r - 2r)$

$A = 0.5\pi r^2 + 2rh$

$= 0.5\pi r^2 + 2r[0.5(12 - \pi r - 2r)]$

$= 0.5\pi r^2 + 12r - \pi r^2 - 2r^2$

Maximum area when $\frac{dA}{dr} = 0$ and $\frac{d^2A}{dr^2} < 0$

$\frac{dA}{dr} = \pi r + 12 - 2\pi r - 4r = 12 - \pi r - 4r$

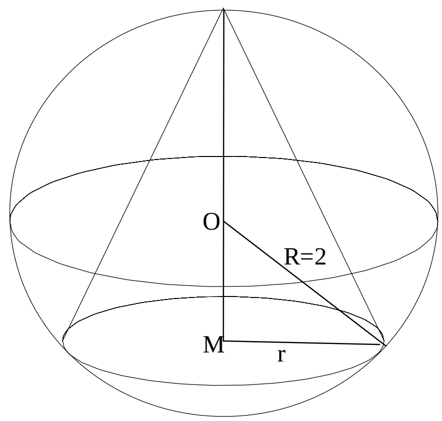
$\frac{d^2A}{dr^2} = -\pi - 4$

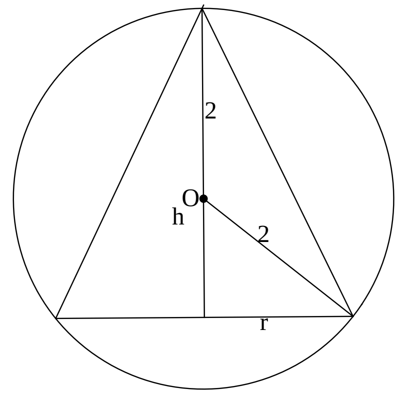
At $\frac{dA}{dr} = 0$, $r(\pi + 4) = 12 \quad \Rightarrow \quad r = 1.68$

At $r = 1.68$, $h = 1.68$

Dimensions are 1.68 m high and 3.36 m wide.

10





$V_{\text{cone}} = \frac{\pi}{3} r^2 h \qquad OM = h - 2 \Rightarrow 4 = r^2 + (h-2)^2$

$V_{\text{cone}} = \frac{\pi}{3} [4 - (h-2)^2]h$

$= \frac{\pi}{3}(4 - h^2 + 4h - 4)h$

$= \frac{\pi}{3}(4h^2 - h^3)$

Maximum volume of cone when $\frac{dV}{dh} = 0$ and $\frac{d^2V}{dh^2} < 0$

$\frac{dV}{dh} = \frac{\pi}{3}(8h - 3h^2)$

$\frac{d^2V}{dh^2} = \frac{\pi}{3}(8 - 6h)$

At $\frac{dV}{dh} = 0$, $8h = 3h^2$ with $h \neq 0$, $h = 2\frac{2}{3} \approx 2.67$

At $h = 2.67$, $\frac{d^2V}{dh^2} < 0$ so maximum volume

$r = ?\quad r^2 = 4 - (h-2)^2 \qquad r = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3} \approx 1.89\,\text{m}$

The dimensions of the cone of greatest volume that can just fit inside a sphere of radius 2 m are $r = \frac{4\sqrt{2}}{3} \approx 1.89\,\text{m}$ and $h = 2\frac{2}{3} \approx 2.67\,\text{m}$.

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11 Perimeter = 6 m

$P = 0.5(2\pi r) + 2h + 2r$

$6 = \pi r + 2h + 2r \Rightarrow h = 0.5(6 - \pi r - 2r)$

$A = 0.5\,\pi r^2 + 2rh$

$= 0.5\,\pi r^2 + 2r[0.5(6 - \pi r - 2r)]$

$= 0.5\,\pi r^2 + 6r - \pi r^2 - 2r^2$

$= -\,0.5\,\pi r^2 + 6r - 2r^2$

Maximum area when $\frac{dA}{dr} = 0$ and $\frac{d^2A}{dr^2} < 0$

$$\frac{dA}{dr} = -\pi r + 6 - 4r$$

$$\frac{d^2A}{dr^2} = -\pi - 4$$

At $\frac{dA}{dr} = 0$, $r(\pi + 4) = 6 \quad \Rightarrow \quad r = \frac{6}{\pi + 4}$

$$A = -\,0.5\pi r^2 + 6r - 2r^2$$

$$A = -0.5\pi\left(\frac{6}{\pi+4}\right)^2 + 6\times\frac{6}{\pi+4} - 2\left(\frac{6}{\pi+4}\right)^2$$

$$A = -0.5\pi\left(\frac{6}{\pi+4}\right)^2 + 6\times\frac{6(\pi+4)}{(\pi+4)^2} - 2\left(\frac{6}{\pi+4}\right)^2$$

$$= \frac{-18\pi + 36\pi + 144 - 72}{(\pi+4)^2}$$

$$= \frac{18\pi + 72}{(\pi+4)^2}$$

$$= \frac{18(\pi+4)}{(\pi+4)^2}$$

$$A = \frac{18}{\pi+4}\,\text{m}^2$$

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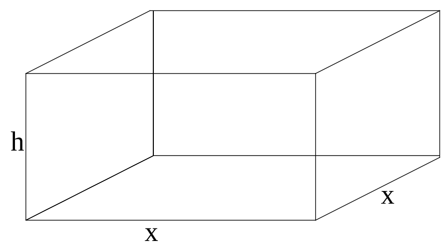
12



$h + x \le 30$ Use 30 for maximum volume.

$V = x^2h$, but $h = 30 - x$

$V = x^2(30 - x)$

$\quad = 30x^2 - x^3$

Maximum volume when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0$

$\frac{dV}{dx} = 60x - 3x^2$

$\frac{d^2V}{dx^2} = -6x$

At $\frac{dV}{dx} = 0$, $60x - 3x^2 = 0$

$3x(20 - x) = 0 \Rightarrow x = 0$ or $x = 20$

At $x = 20$, $\frac{d^2V}{dx^2} < 0$ so maximum volume

Max V is 4000 cm^3.

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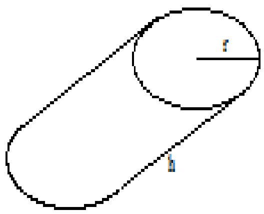
13 $h + C \le 150$ cm

$h + 2\pi r \le 150$ cm



Max volume = ?

$V = \pi r^2$

$V = \pi r^2(150 - 2\pi r)$

$= 150\pi r^2 - 2\pi^2 r^3$

Maximum volume when $\frac{dV}{dr} = 0$ and $\frac{d^2V}{dr^2} < 0$

$\frac{dV}{dr} = 300\pi r - 6\pi^2 r^2$

$\frac{d^2V}{dr^2} = 300\pi - 12\pi^2 r$

At $\frac{dV}{dr} = 0$, $300\pi r = 6\pi^2 r^2 \quad (r \ne 0)$

$50 = \pi r \quad \Rightarrow \quad r = \frac{50}{\pi}$

At $r = \frac{50}{\pi}$, $\frac{d^2V}{dr^2} = 300\pi - 12\pi^2\left(\frac{50}{\pi}\right) < 0$ so maximum volume

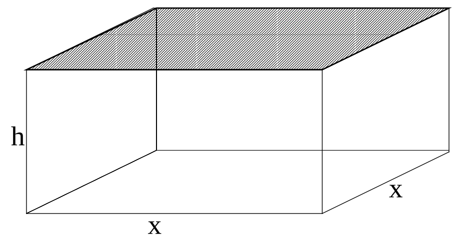
$h = 150 - 2\pi r$

$= 150 - 2\pi\left(\frac{50}{\pi}\right)$

$= 50$

The dimensions of the cylinder with the largest volume are $h = 50$ cm and $r = \frac{50}{\pi}$ cm.

14 $C = x^2 + 4xh + 2x^2$ and $x^2h = 324$



$C = 3x^2 + 4x \times \frac{324}{x^2}$

$= 3x^2 + \frac{1296}{x}$

Want minimum cost C

Minimum cost when $\frac{dC}{dx} = 0$ and $\frac{d^2C}{dx^2} > 0$

$\frac{dC}{dx} = 6x - \frac{1296}{x^2}$

$\frac{d^2C}{dx^2} = 6 + \frac{2592}{x^3}$

At $\frac{dC}{dx} = 0$, $6x = \frac{1296}{x^2}$

$x^3 = 216$

$x = 6$

At $x = 6$, $\frac{d^2C}{dx^2} > 0$ so minimum volume

$h = \frac{324}{x^2} = 9$

For minimum volume, the dimensions are square base of side 6 m and height of 9 m.

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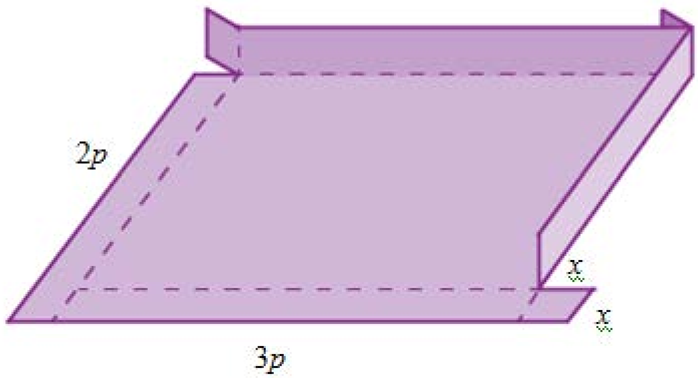
15



$V = x(2p - 2x)(3p - 2x)$, where length of cardboard = $3p$

$= 4x^3 - 10x^2p + 6xp^2$ Can do for the general case and then substitute values of p.

$$\frac{dV}{dx} = 12x^2 - 20xp + 6p^2 = 2(6x^2 - 10xp + 3p^2)$$

$$\frac{dV}{dx} = 0 \text{ for } x = \frac{10p \pm \sqrt{100p^2 - 4 \times 6 \times 3p^2}}{12}$$

$$= \frac{10p \pm \sqrt{100p^2 - 72p^2}}{12}$$

$$= \frac{10p \pm p\sqrt{28}}{12} = \frac{2p(5 \pm \sqrt{7})}{12} = \frac{p(5 \pm \sqrt{7})}{6}$$

But $2p - 2x > 0$, so only $x = \dfrac{p(5 - \sqrt{7})}{6}$ is possible.

$$\frac{d^2V}{dx^2} = 24x - 20p \text{ and for } x = \frac{p(5-\sqrt{7})}{6},\ \frac{d^2V}{dx^2} = 24 \times \frac{p(5-\sqrt{7})}{6} - 20p$$

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Now $24 \times \dfrac{p(5-\sqrt{7})}{6} - 20p = p(20 - 4\sqrt{7}) - 20p < 0$ so there is a maximum

at $\dfrac{p(5-\sqrt{7})}{6}$.

The dimensions are $3p - 2x$, $2p - 2x$ and x.

a For length 20 cm, these are 14.77 cm × 8.10 cm × 2.62 cm

b For length 30 cm, these are 22.15 cm × 12.15 cm × 3.92 cm

c For length 50 cm, these are 36.92 cm × 20.25 cm × 6.54 cm

Or doing individually:

a $3p = 20 \Rightarrow 2p = \dfrac{40}{3}$

$$V = x\left(\frac{40}{3} - 2x\right)(20 - 2x) = \frac{800x}{3} - 40x^2 - \frac{80x^2}{3} + 4x^3$$

Maximum volume when $\dfrac{dV}{dx} = 0$ and $\dfrac{d^2V}{dx^2} < 0$

$$\frac{dV}{dx} = \frac{800}{3} - 80x - \frac{160x}{3} + 12x^2$$

$$\frac{d^2V}{dx^2} = -80 - \frac{160}{3} + 24x$$

At $\dfrac{dV}{dx} = 0$, $\dfrac{800}{3} - 80x - \dfrac{160x}{3} + 12x^2 = 0$

$800 - 400x + 36x^2 = 0 \Rightarrow x = 2.62$ cm, 8.50 too big

At $x = 2.62$, $\dfrac{d^2V}{dx^2} < 0$ so maximum volume

$3p = 20$ Length of box is 14.77 cm

$2p = \dfrac{40}{3} = 13.33$ Width of tray is 8.10 cm

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b $3p = 30 \Rightarrow 2p = 20$

$$V = x(30-2x)(20-2x) = 600x - 100x^2 + 4x^3$$

Maximum volume when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0$

$$\frac{dV}{dx} = 600 - 200x + 12x^2$$

$$\frac{d^2V}{dx^2} = -200 + 24x$$

At $\frac{dV}{dx} = 0$, $600 - 200x + 12x^2 = 0$

$x = 3.92\,\text{cm}$

At $x = 3.92$, $\frac{d^2V}{dx^2} < 0$ so maximum volume

$3p = 30$ Length of tray is 22.15 cm

$2p = 20$ Width of tray is 12.15 cm

c $3p = 50 \Rightarrow 2p = \frac{100}{3}$

$$V = x(50 - 2x)\left(\frac{100}{3} - 2x\right) = \frac{5000x}{3} - \frac{200}{3}x^2 - 100x^2 + 4x^3$$

Maximum volume when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2} < 0$

$$\frac{dV}{dx} = \frac{5000}{3} - \frac{400}{3}x - 200x + 12x^2$$

$$\frac{d^2V}{dx^2} = -\frac{400}{3} - 200 + 24x$$

At $\frac{dV}{dx} = 0$, $\frac{5000}{3} - \frac{400}{3}x - 200x + 12x^2 = 0$

$x = 6.54\,\text{cm}$

At $x = 6.54$, $\frac{d^2V}{dx^2} < 0$ so maximum volume

$3p = 50$ Length of tray is 36.92 cm

$2p = 33.33$ Width of tray is 20.25 cm

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Exercise 3.08 Optimisation in business

Reasoning and communication

1 $C(x) = 2000 - 75x - 5x^2 + \frac{x^3}{3}$

Costs are minimised when $C'(x) = 0$ and $C''(x) > 0$.

$C'(x) = -75 - 10x + x^2$

$C''(x) = -10 + 2x$

If $C'(x) = 0$, then $-75 - 10x + x^2 = 0 \Rightarrow (x - 15)(x + 5) = 0$

$x > 0$ so $x = 15$

$C''(15) = -10 + 30 > 0$ so minimum.

For minimum cost, the production level is 15 tonnes of silver.

2 Let x be the number of machines used.

$$N = x\left(30 - \frac{x^2}{10}\right) = 30x - \frac{x^3}{10}$$

Maximum production when $N'(x) = 0$ and $N''(x) < 0$.

$$N'(x) = 30 - \frac{3x^2}{10}$$

$$N''(x) = -\frac{6x}{10} < 0 \text{ for } x > 0$$

At $N'(x) = 0$, $30 = \frac{3x^2}{10}$

$x = 10$

At $x = 10$, $N''(x) < 0$ so maximum production.

Ten additional machines, so 11 machines should be used to achieve

the maximum production.

3 **a** $P(x) = 4x - (50 + 1.3x + 0.001x^2)$

$= -0.001x^2 + 2.7x - 50$

b Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.

$P'(x) = -0.002x^2 + 2.7$
$P''(x) = -0.004x < 0 \text{ for } x > 0$
$\text{At } P'(x) = 0, \quad 0.002x^2 = 2.7$
$x = 1350$
$\text{At } x = 1350, \ P''(x) < 0 \text{ so maximum profit.}$

Produce 1350 items for maximum profit.

4 **a** $R(x) = (5 - 0.001x)x$

$= 5x - 0.001x^2$

Maximum revenue when $R'(x) = 0$ and $R''(x) < 0$.

$R'(x) = 5 - 0.002x$
$R''(x) = -0.002 < 0$
$\text{At } R'(x) = 0, \quad 0.002x = 5 \Rightarrow x = 2500$
$x = 2500$
$\text{At } x = 2500, R''(x) < 0 \text{ so maximum revenue.}$

b $P(x) = (5 - 0.001x)x - (2800 + x)$

$= 4x - 0.001x^2 - 2800$

Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.

$P'(x) = 4 - 0.002x$
$P''(x) = -0.002 < 0$
$\text{At } P'(x) = 0, \quad 0.002x^2 = 4$
$x = 2000$
$\text{At } x = 2000, \ P''(x) < 0 \text{ so maximum profit.}$

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5 **a** $C(x) = 4000 - 3x + 10^{-3}x^2$

Minimum cost when $C'(x) = 0$ and $C''(x) > 0$.

$C'(x) = -3 + 0.002x$
$C''(x) = 0.002 > 0$
At $C'(x) = 0, \quad 0.002x = 3$
$x = 1500$
At $x = 1500$, $C''(x) > 0$ so minimum cost.

b $P(x) = 4x - (4000 - 3x + 10^{-3}x^2)$

$= 7x - 0.001x^2 - 4000$

Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.

$P'(x) = 7 - 0.002x$
$P''(x) = -0.002 < 0$
At $P'(x) = 0, \quad 0.002x = 7$
$x = 3500$
At $x = 3500$, $P''(x) < 0$ so maximum profit.

Maximum profit is \$8250.

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6 $P(x) = (d)x - \left(600 + 10x + \frac{1}{2}x^2\right)$ where $x = 600 - 3d$

$$P(x) = \left(200 - \frac{x}{3}\right)x - \left(600 + 10x + \frac{1}{2}x^2\right)$$
$$= -\frac{5x^2}{6} + 190x - 600$$

Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.

$$P'(x) = -\frac{5x}{3} + 190$$
$$P''(x) = -\frac{5}{3} < 0$$

At $P'(x) = 0$, $\frac{5x}{3} = 190$

$x = 114$

At $x = 114$, $P''(x) < 0$ so maximum profit.

To maximise profits, he should sell 114 clock radios at $162.

7 **a** $C(x) = 40\,000 - 30x + 10^{-2}x^2$

Minimum cost when $C'(x) = 0$ and $C''(x) > 0$.

$$C'(x) = -30 + 10^{-2} \times 2x$$
$$C''(x) = 10^{-2} \times 2$$

At $C'(x) = 0$, $15 = 10^{-2}x$

$x = 1500$

At $x = 1500$, $C''(x) > 0$ so minimum cost.

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b $P(x) = 40x - (40\,000 - 30x + 0.01x^2)$

$= -0.01x^2 + 70x - 40\,000$

Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.

$P'(x) = -0.02x + 70$
$P''(x) = -0.02 < 0$
At $P'(x) = 0$, $x = 3500$
At $x = 3500$, $P''(x) < 0$ so maximum profit.

Maximum profit is \$82 500.

8 $C(x) = 100 + 28x - 5x^2 + \frac{x^3}{3}$, $p = 5000 - 5x$ per roll of carpet

a $TC(x) = 100 - 5x^2 + \frac{x^3}{3} + 250x$

b $R(x) = (5000 - 5x)x$

c $P(x) = R(x) - TC(x)$

$$P(x) = (5000 - 5x)x - \left(100 + 5x^2 + \frac{x^3}{3} + 250x\right)$$
$$= -\frac{x^3}{3} + 4750x - 100$$

d $P(x) = -\frac{x^3}{3} + 4750x - 100$

Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.
$P'(x) = -x^2 + 4750$
$P''(x) = -2x$
At $P'(x) = 0$, $x = 69.12$
At $x = 69$, $P''(x) < 0$ so maximum profit.

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9 $L\left(x\right)=-\frac{5000}{n+1}-80n$

$L'\left(x\right)=\frac{5000}{\left(n+1\right)^2}-80$

$L''\left(x\right)=-\frac{5000}{\left(n+1\right)^3}$

Minimum loss when $L'(x)=0$ and $L''(x)>0$.

$\frac{5000}{\left(n+1\right)^2}=80$

$\left(n+1\right)^2=62.5 \Rightarrow n=6.9$

$L''(x)>0$ so minimum.

$n = 7$

10 $C(v) = F(v) + D(v)$

$C\left(v\right) = 2v^{\frac{3}{2}}+59+\frac{1.5\times 10^5}{v}+2\,000$

$C'\left(x\right)=3v^{\frac{1}{2}}-\frac{1.5\times 10^5}{v^2}$

$C''\left(x\right)=\frac{3}{2\sqrt{v}}+\frac{1.5\times 10^5}{v^3}>0$ for $v>0$

Minimum cost when $C'(x)=0$ and $C''(x)>0$.

$3\sqrt{v}=\frac{1.5\times 10^5}{v^2}$

$v=75.79$

$C''(x)>0$ so minimum.

$C(75.79) = 5357.66$ cents

Cost is \$53.58.

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Exercise 3.09 General optimisation problems

Reasoning and communication

1 $h = 40t - 5t^2 + 4$

Maximum height h when $\frac{dh}{dt} = 0$ and $\frac{d^2h}{dt^2} < 0$.

$\frac{dh}{dt} = 40 - 10t$

At $\frac{dh}{dt} = 0$, $t = 4$ max or min?

$\frac{d^2h}{dt^2} = -10 < 0$ so maximum at $t = 4$

At $t = 4$, $h = 84$

The projectile reaches 84 m in height.

2 $x = 2 + 3t - t^2$

$\frac{dx}{dt} = 3 - 2t$

$\frac{d^2x}{dt^2} = -2$

Maximum displacement when $\frac{dx}{dt} = 0$ and when $\frac{d^2x}{dt^2} < 0$.

$t = 1.5$

$x = 4.25$ cm

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3 $x = 15t - 3t^2$

$$\frac{dx}{dt} = 15 - 6t$$

$$\frac{d^2x}{dt^2} = -6$$

Maximum displacement when $\frac{dx}{dt} = 0$ and when $\frac{d^2x}{dt^2} < 0$.

$t = 2.5$

$x = 18.75$ m

The ball will go 18.75 m up the slope before it starts to roll back down.

4 $h = -4t^2 + 4t + 10$

Maximum height when $\frac{dh}{dt} = 0$ and when $\frac{d^2h}{dt^2} < 0$.

$$\frac{dh}{dt} = -8t + 4$$

$$\frac{d^2h}{dt^2} = -8$$

At $\frac{dh}{dt} = 0$, $t = 0.5$

At $t = 0.5$, $\frac{d^2h}{dt^2} < 0$ so maximum height.

$h = 11$

The maximum height she attains is 11 m.

5 $h = 8 - 8t^2 + 32t$

Maximum height when $\frac{dh}{dt} = 0$ and when $\frac{d^2h}{dt^2} < 0$.

$$\frac{dh}{dt} = -16t + 32$$
$$\frac{d^2h}{dt^2} = -16$$

At $\frac{dh}{dt} = 0$, $t = 2$

At $t = 2$, $\frac{d^2h}{dt^2} < 0$ so maximum height.

$h = 40$

The maximum height reached by the rocket is 40 m.

6 $y = -5t^2 + 8t + 35$

Maximum height when $\frac{dh}{dt} = 0$ and when $\frac{d^2h}{dt^2} < 0$.

$$\frac{dy}{dt} = -10t + 8$$
$$\frac{d^2y}{dt^2} = -10$$

At $\frac{dy}{dt} = 0$, $t = 0.8$

At $t = 0.8$, $\frac{d^2y}{dt^2} < 0$ so maximum height.

$y = 38.2$

The maximum height it reaches is 38.2 m.

7 $P = -2\frac{2}{3}T^3 + 52T^2 - 176T + 382$

Maximum population when $\frac{dP}{dT} = 0$ and $\frac{d^2P}{dT^2} < 0.$

Minimum population when $\frac{dP}{dT} = 0$ and $\frac{d^2P}{dT^2} > 0.$

$\frac{dP}{dT} = -8T^2 + 104T - 176$

$\frac{d^2P}{dT^2} = -16T + 104$

At $\frac{dP}{dT} = 0$, $-8T^2 + 104T - 176 = 0$

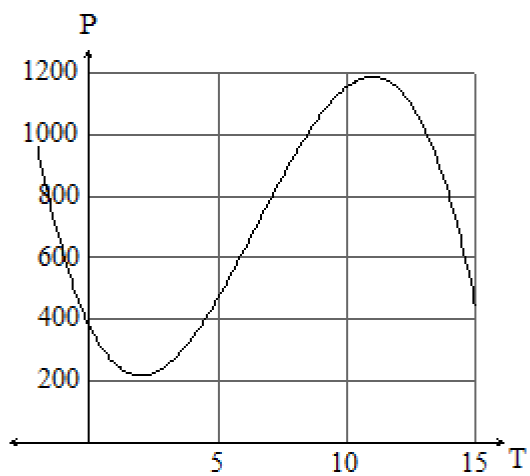
$T^2 - 13T + 22 = 0$

$(T - 11)(T - 2) = 0$

$T = 11$ or $T = 2$

At $T = 11$, $\frac{d^2P}{dT^2} < 0$ so maximum population of $P = 1189$.

At $T = 2$, $\frac{d^2P}{dT^2} > 0$ so minimum population of $P = 217$.



The diagram shows that the maximum and minimum populations do not occur at the end points.

The minimum population is 217 and occurs at a temperature of 2°C.

The maximum population is 1189 and occurs at a temperature of 11°C.

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8

Number of trees	Number of kg	Yield
75	7	525
$75-1$	$7+0.2$	$(75-1)(7+0.2)$
$75-2$	$7+(0.2)2$	$(75-2)(7+2(0.2))$
…	…	…
$75-x$	$7+(0.2)x$	$(75-x)(7+x(0.2))$

$$Y=(75-x)(7+0.2x)$$
$$=525+8x-0.2x^2$$

Maximum Y when $\frac{dY}{dx}=0$ and $\frac{d^2Y}{dx^2}<0$.

$$\frac{dY}{dx}=8-0.4x$$

$$\frac{d^2Y}{dx^2}=-0.4<0 \text{ for } x>0$$

At $\frac{dY}{dx}=0$, $x=20$

$$\frac{d^2Y}{dx^2}=-2<0 \text{ so maximum yield.}$$

The farmer should plant 55 (75 – 20) trees per hectare to maximise the total yield.

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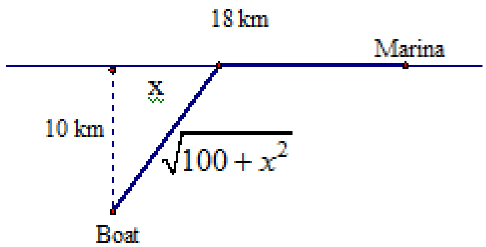
9 $T = T_{row} + T_{run}$, $\text{time} = \dfrac{\text{distance}}{\text{velocity}}$



$$T = \frac{\sqrt{100+x^2}}{6} + \frac{18-x}{8}$$

Minimum time T when $\dfrac{dT}{dx} = 0$ and $\dfrac{d^2T}{dx^2} > 0.$

$$\frac{dT}{dx} = \frac{1}{6}\left[\frac{1}{2}\left(100+x^2\right)^{-\frac{1}{2}}(2x)\right] - \frac{1}{8}$$

$$\frac{dT}{dx} = \frac{1}{6}\left(\frac{2x}{2\sqrt{100+x^2}}\right) - \frac{1}{8}$$

When $\dfrac{dT}{dx} = 0, \dfrac{x}{6\sqrt{100+x^2}} = \dfrac{1}{8}$

$4x = 3\sqrt{100+x^2}$

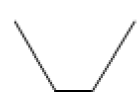
$900 + 9x^2 = 16x^2$

$7x^2 = 900$

$x > 0$

$x = 11.34$

x	0	11	12
$\dfrac{dp}{dt}$	−	0	+



Concave up so minimum at $x = 11.34$

He should land 11.34 km towards the marina to get there in minimum time.

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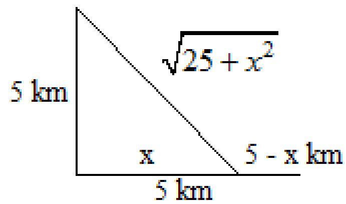
10



$T = T_{track} + T_{road}$ $\qquad$ $\text{time} = \frac{\text{distance}}{\text{velocity}}$

$$T = \frac{\sqrt{25+x^2}}{3} + \frac{5-x}{5}$$

Minimum time T when $\frac{dT}{dx} = 0$ and $\frac{d^2T}{dx^2} > 0$.

$$\frac{dT}{dx} = \frac{1}{3}\left[\frac{1}{2}\left(25+x^2\right)^{-\frac{1}{2}}(2x)\right] - \frac{1}{5}$$

$$= \frac{1}{3}\left(\frac{2x}{2\sqrt{25+x^2}}\right) - \frac{1}{5}$$

When $\frac{dT}{dx} = 0, \frac{x}{3\sqrt{25+x^2}} = \frac{1}{5}$

$5x = 3\sqrt{25+x^2}$

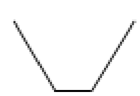
$225 + 9x^2 = 25x^2$

$16x^2 = 225$

$x > 0$

$x = 3.75\,\text{km}$

x	0	3.75	4
$\frac{dT}{dx}$	–	0	+



Concave up so minimum at $x = 3.75$ km.

He should run down the very windy track until he reaches the point that is closest to the road, then head across country at an angle of 36.9°, from the perpendicular, and then he can cross-country for 3.75 km, reach the road and run the rest.

The shortest possible time is $T = 2.33$ i.e. two hours 20 minutes.

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Chapter 3 Review

Multiple choice

1 C Let $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Let $x = 144$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{144}} = \frac{1}{24}$

$\delta x = 3$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx} \times \delta x \\ &= \frac{1}{24} \times 3 \\ &= \frac{1}{8}\end{aligned}$$

$$\sqrt{147} \approx 12 + \frac{1}{8} = 12\frac{1}{8} = 12.125$$

2 B $y = 3\cos(4x)$

$$\frac{dy}{dx} = -12\sin(4x)$$

$$\frac{d^2y}{dx^2} = -48\cos(4x)$$

3 A Gradient is negative and $f''(-1) < 0$ so concave downwards.

4 C Let x and y be the two numbers. $x - y = 24$

Let $R = xy + x + y$. Want minimum R.

$R = x(x - 24) + x + x - 24$

$= x^2 - 22x - 24$

Minimum R when $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$.

$\frac{dR}{dx} = 2x - 22$

$\frac{d^2R}{dx^2} = 2$

At $\frac{dR}{dx} = 0$, $x = 11$

$\frac{d^2R}{dx^2} = 2 > 0$ so minimum.

$x = 11$, $y = ?$

$x - y = 24 \Rightarrow y = -13$

The two numbers are 11 and –13.

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5 D Let x and y be the sides of the rectangle. $2x + 2y = 80$.

$A = xy$

$A = x(40 - x) = 40x - x^2$

Maximum A when $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$.

$\frac{dA}{dx} = 40 - 2x$

$\frac{d^2A}{dx^2} = -2$

At $\frac{dA}{dx} = 0$, $x = 20$

$\frac{d^2A}{dx^2} = -2 < 0$ so maximum.

$x = 20$, $y = ?$

$x + y = 40 \Rightarrow y = 20$

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Short answer

6 **a** Estimate $33^{0.4}$

Let $y = x^{\frac{2}{5}}$

$$\frac{dy}{dx} = \frac{2}{5}x^{-\frac{3}{5}} = \frac{2}{5\sqrt[5]{x^3}}$$

Let $x = 32$, then $\frac{dy}{dx} = \frac{2}{5\sqrt[5]{32^3}} = \frac{2}{5\times 2^3} = \frac{1}{20}$

$\delta x = 1$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times\delta x\\ &= \frac{1}{20}\times 1\\ &= \frac{1}{20}\end{aligned}$$

$33^{0.4} \approx 4 + \frac{1}{20} = 4.05$

b Estimate $31.7^{0.4}$

$\delta x = -0.3$

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}\times\delta x\\ &= \frac{1}{20}\times(-0.3)\\ &= -\frac{3}{200}\end{aligned}$$

$33^{0.4} \approx 4 - \frac{3}{200} = 4 - 0.015$

$33^{0.4} \approx 3.985$

7 $V = \dfrac{4\pi r^3}{3}$ and $\dfrac{dr}{dt} = 0.04\,\text{cm/sec}$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

At $r = 6\,\text{cm}$

$\dfrac{dV}{dt} = 4\pi \times 6^2\left(0.04\right) = 18.096\ \text{cm}^3/\text{sec}$

8 **a** I L = 1000 cm^3

height : diameter = 1.6 : 1 $\Rightarrow h = 1.6(2r)$

$h = 3.2r$

$\pi r^2 h = 60\,000 \Rightarrow \pi r^2(3.2r) = 60\,000$

$r^3 = \dfrac{60\,000}{3.2\pi}$

$r = 18.14$ cm, $h = 3.2r \Rightarrow h = 58.05$ cm

b $V = \pi r^2 x$

$\dfrac{dV}{dx} = \pi r^2 \times 1 = \pi r^2$

$\delta V \approx \dfrac{dV}{dx} \times \delta h$

$$\begin{aligned}\frac{\delta V}{V} \times 100\% &= \pi r^2 \times \frac{1}{V} \times \delta x \times 100\% \\ &= \pi r^2 \times \frac{1}{\pi r^2 x} \times \delta x \times 100\% \\ &= \frac{\delta x}{x} \times 100\%\end{aligned}$$

$\therefore$ The approximate percentage error in volume is x%.

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c $V=\pi\left(\frac{y}{2}\right)^2 x$

$$\frac{dV}{dy}=2\pi\times\frac{y}{4}\times x=\pi r^2$$

$$\delta V\approx\frac{dV}{dy}\times\delta y$$

$$\frac{\delta V}{V}\times 100\%=\pi\times\frac{y}{2}x\times\frac{1}{V}\times\delta y\times 100\%$$

$$=\cancel{\pi}\frac{y}{2}\cancel{x}\times\frac{1}{\cancel{\pi}\left(\frac{y}{2}\right)^2\cancel{x}}\times\delta d\times 100\%$$

$$=2\frac{\delta y}{y}\times 100\%$$

$\therefore$ The approximate percentage error in volume is $2y\%$.

d $V=\pi\left(\frac{1.03y}{2}\right)^2 1.03x$

$$V=\pi\left(\frac{y}{2}\right)^2 x\times 1.03^3=V\times 1.09$$

Increase of 9%, so 9% error in volume.

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9 **a** $y = 5x^6 - 3x^2 + x + 10$

$$\frac{dy}{dx} = 30x^5 - 6x + 1$$

$$\frac{d^2y}{dx^2} = 150x^4 - 6$$

b $f(t) = (2t + 9)^4$

$$f'(t) = 4(2t+9)^3 \times 2$$

$$= 8(2t+9)^3$$

$$f''(t) = 24(2t+9)^2 \times 2$$

$$= 48(2t+9)^2$$

c $f(n) = (3n - 1)^2(2n + 4)$

$$f'(n) = 2(3n-1)\times 3\times(2n+4) + 2(3n-1)^2$$

$$= 2(3n-1)(6n+12+3n-1)$$

$$= 2(3n-1)(9n+11)$$

$$f''(n) = 2\left[3(9n+11) + 9(3n-1)\right]$$

$$= 2(54n+24) = 4(27n+12)$$

d $y = \dfrac{6x-9}{3x-1}$

$$\frac{dy}{dx} = \frac{6(3x-1) - 3(6x-9)}{(3x-1)^2}$$

$$= \frac{21}{(3x-1)^2}$$

$$\frac{d^2y}{dx^2} = -42(3x-1)^{-3} \times 3$$

$$= \frac{-126}{(3x-1)^3}$$

10 **a** $x = t^3 - 12t^2 + 36t - 9$ cm

$v = 3t^2 - 24t + 36$ cm/s

b $a = 6t - 24$ cm/s^2

c $a = -12$ cm/s^2

11 $y = 2x^3 - 7x^2 - 3x + 1$

$$\frac{dy}{dx} = 6x^2 - 14x - 3$$

$$\frac{d^2y}{dx^2} = 12x - 14$$

Concave upwards for $\frac{d^2y}{dx^2} > 0$.

$$12x - 14 > 0$$

$$x > \frac{7}{6}$$

12 $f(x) = 4x^7$

$f'(x) = 28x^6$

$f''(x) = 168x^5$

If $f''(x) = 0$, then $x = 0$.

x	-1	0	1
$f''(x)$	–	0	+

For $x < 0$, concave downwards; for $x > 0$, concave upwards.

Yes, the function has a point of inflection at (0, 0).

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13 $y = x^4 + 4x^3 - 48x^2 + 1$

$$\frac{dy}{dx} = 4x^3 + 12x^2 - 96x$$
$$\frac{d^2y}{dx^2} = 12x^2 + 24x \ \ - 96$$

Points of inflection occur where $\frac{d^2y}{dx^2} = 0.$

$0 = 12(x^2 + 2x - 8)$

$0 = (x + 4)(x - 2)$

$x = -4$ or $x = 2$

Points of inflection are (–4, –767), (2, –143).

14 **a** $y = 3x^2 - 6x + 3$

$$\frac{dy}{dx} = 6x - 6$$
$$\frac{d^2y}{dx^2} = 6 > 0 \text{ for all values of } x$$

Stationary point where $\frac{dy}{dx} = 0$ i.e. (1, 0)

$\frac{d^2y}{dx^2} > 0$ so concave up. Minimum turning point.

b $f(x) = 5x - x^2$

$$f'(x) = 5 - 2x$$
$$f''(x) = -2 < 0 \text{ for all values of } x$$

Stationary point where $\frac{dy}{dx} = 0$ i.e. (2.5, 6.25).

$\frac{d^2y}{dx^2} < 0$ so concave down. Maximum turning point.

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c $f(x) = 3x^4 - 4x^3 - 12x^2 + 7$

$f'(x) = 12x^3 - 12x^2 - 24x$

$f''(x) = 6x^2 - 24x - 24$

Stationary point where $\frac{dy}{dx} = 0$

$12x\left(x^2 - x - 2\right) = 0$

$x(x-2)(x+1) = 0$

$x = 0,\ 2, -1$

At $x = 2$, $f''(x) > 0$ so concave up. Minimum turning point at $(2, -25)$.

At $x = 0$, $f''(x) < 0$ so concave down. Maximum turning point at $(0, 7)$.

At $x = -1$, $f''(x) > 0$ so concave up. Minimum turning point at $(-1,\ 2)$.

d $y = (2x - 1)^4$

$f'(x) = 4(2x-1)^3 \times 2 = 8(2x-1)^3$

$f''(x) = 48(2x-1)^2$

Stationary point where $f'(x) = 0$.

$8(2x-1)^3 = 0$

$x = \frac{1}{2}$

At $x = \frac{1}{2}$, $f''(x) > 0$ so concave up. Minimum turning point at $\left(\frac{1}{2},\ 0\right)$.

As the power of the x is even, there is no point of inflection.

15 $y = 2x^3 - 1$

$$\frac{dy}{dx} = 6x^2$$

$$\frac{d^2y}{dx^2} = 12x$$

Stationary point where $\frac{dy}{dx} = 0$, i.e. $(0, -1)$.

Check

$$\frac{d^2y}{dx^2} = 0 \text{ at } x = 0$$

So stationary point of inflection, not a turning point.

16 **a** $\frac{dP}{dt} > 0$ and $\frac{d^2P}{dt^2} < 0$

b The number of possums is increasing, but at a decreasing rate.

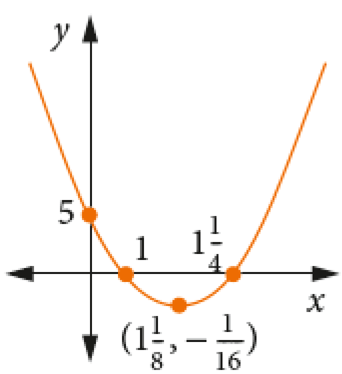
c The rate of population growth is decreasing.

17 **a** $y = 4x^2 - 9x + 5$

$$\frac{dy}{dx} = 8x - 9$$

$\frac{d^2y}{dx^2} = 8 > 0$, so minimum turning point.

Stationary point where $\frac{dy}{dx} = 0$, i.e. (1.125, 0).



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b $f(x) = 2x^3 - 6x$

$f'(x) = 6x^2 - 6$

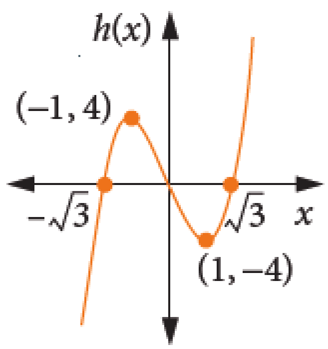
$f''(x) = 12x$

Stationary points where $\frac{dy}{dx} = 0$, i.e. $(1, -4), (-1, 4)$.

$f''(1) > 0$, so minimum turning point.

$f''(-1) < 0$, so maximum turning point.

Point of inflection at $f''(x) = 0$, i.e. at $(0, 0)$.



c $f(x) = 2x^3 - 12x^2$

$f'(x) = 6x^2 - 24x$

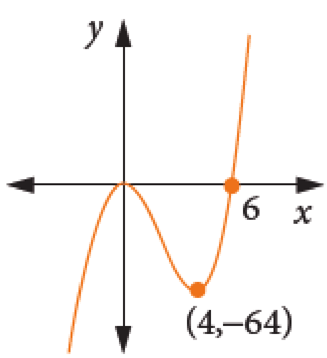
$f''(x) = 12x - 24$

Stationary points where $f'(x) = 0$, i.e. $(0, 0)$, $(4, -64)$.

$f''(4) > 0$, so minimum turning point.

$f''(0) < 0$, so maximum turning point.

Point of inflection at $f''(x) = 0$, i.e. at $(2, -32)$.



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d $f(x) = x^3 - 2x^2 + x - 2$

$f'(x) = 3x^2 - 4x + 1$

$f''(x) = 6x - 4$

Stationary points where $f'(x) = 0$.

$3x^2 - 4x + 1 = 0$

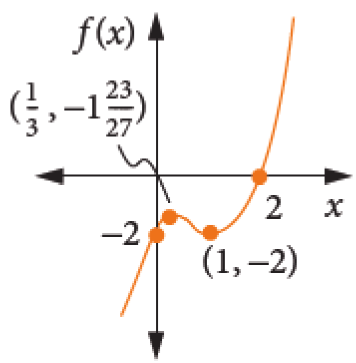
$(3x - 1)(x - 1) = 0$

i.e. $(1, -2), \left(\frac{1}{3}, -1.85\right)$

$f''(1) > 0$, so minimum turning point.

$f''\left(\frac{1}{3}\right) < 0$, so maximum turning point.

Point of inflection at $f''(x) = 0$, i.e. at $\left(\frac{2}{3}, -1.93\right)$.



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e $f(x) = 3x^3 - x^2 - 7x - 3$

$f'(x) = 9x^2 - 2x - 7$

$f''(x) = 18x - 2$

Stationary points where $f'(x) = 0$.

$9x^2 - 2x - 7 = 0$

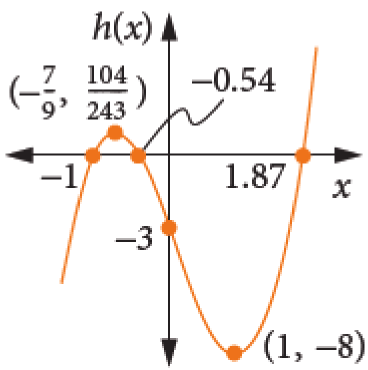
$(9x + 7)(x - 1) = 0$

i.e. $(-0.8, 0.4), (1, -8)$

$f''(1) > 0$, so minimum turning point.

$f''(-0.8) < 0$, so maximum turning point.

Point of inflection at $f''(x) = 0$, i.e. at $(0.7, -5.7)$.



18 Let $S = 3x + y$ and $xy = 48$.

$$\therefore \quad S = 3x + \frac{48}{x}$$

Minimum S when $\frac{dS}{dx} = 0$ and $\frac{d^2S}{dx^2} < 0.$

$$\frac{dS}{dx} = 3 - \frac{48}{x^2}$$

$$\frac{d^2S}{dx^2} = 2 \times \frac{48}{x^3}$$

At $\frac{dS}{dx} = 0$, $3 = \frac{48}{x^2}$

$x = 4$

$\frac{d^2S}{dx^2} = 2 > 0$ so minimum.

$x = 4$, $y = ?$

$y = 12$

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19

Number of trees	Number of fruit	Total output
20	240	4800
$20 + 1$	$240 - 10$	$(20 + 1)(240 - 10) = 4378$
$20 + 2$	$240 - 2 \times 10$	$(20 + 2)(240 - 2 \times 10) = 3960$
$\cdots$	$\cdots$	$\cdots$
$20 + x$	$240 - x \times 10$	$(20 + x)(240 - x \times 10)$

Total ouput = $(20 + x)(240 - 10x)$

Let $T = 10(20 + x)(24 - x) = 10(480 + 4x - x^2)$

Maximum output when $T'(x) = 0$ and $T''(x) < 0$.

$T'(x) = 10(4 - 2x)$

$T''(x) = -20$

At $T'(x) = 0$, $x = 2$

At $x = 2$, $T''(x) < 0$, so maximum output.

$x = 2$

Add 2 trees.

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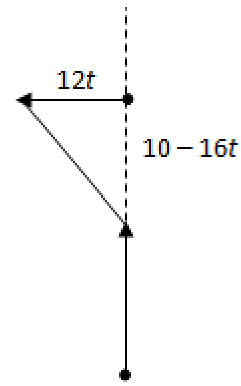
20 **a**



Distance apart

$d^2 = (12t)^2 + (10 - 16t)^2$

$d^2 = 144t^2 + 100 - 320t + 256t^2$

$d^2 = 400t^2 - 320t + 100$

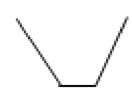
Minimum distance apart when d^2 is a minimum.

$$\frac{dd^2}{dt} = 800t - 320$$

At $\frac{dd^2}{dt} = 0,\ t = \frac{320}{800} = 0.4 \Rightarrow$ 24 minutes after 8 a.m.

Using the sign test

t	0	0.4	1
$\frac{dd}{dt}$	–	0	+



Therefore minimum at $t = 0.4$ h. Closest at 8.24 a.m.

b $d^2 = 400t^2 + 240t + 100$

$d = 6$ km

Minimum distance apart is 6 km.

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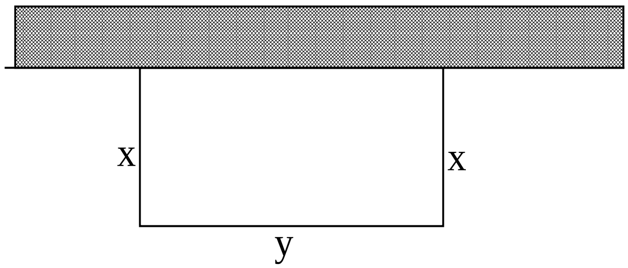

$2x + y = 640$

$A = xy = x(640 - 2x)$

$A = 640x - 2x^2$

Maximum A when $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2} < 0$.

$\frac{dA}{dx} = 640 - 4x$

$\frac{d^2y}{dx^2} = -4$

At $\frac{dy}{dx} = 0$, $x = 160$

At $x = 160$, $\frac{d^2y}{dx^2} < 0$, so maximum area at (160, 320).

The dimensions of the maximum block are 160 m × 320 m.

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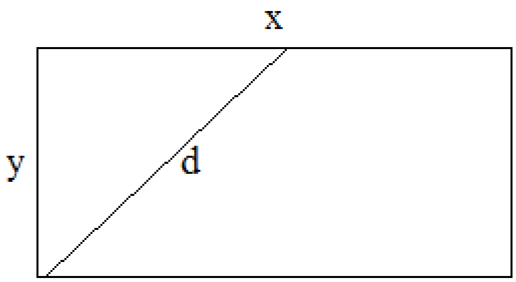
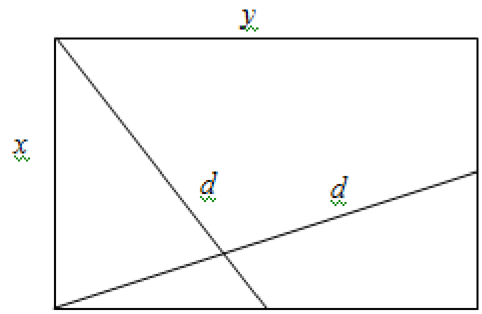
22 $xy = 128$





$d^2 = y^2 + \left(\frac{x}{2}\right)^2$ or $d^2 = x^2 + \left(\frac{y}{2}\right)^2$.

Since it is symmetrical, it doesn't matter which is chosen.

$$d^2 = \left(\frac{128}{x}\right)^2 + \left(\frac{x}{2}\right)^2$$

$$d = \sqrt{\frac{16384}{x^2} + \frac{x^2}{4}}$$

Minimum distance when $\frac{dd}{dx} = 0$ and $\frac{d^2 d}{dx^2} > 0$.

$$\frac{dd}{dx} = \frac{1}{2}\left(\frac{16384}{x^2} + \frac{x^2}{4}\right)^{-\frac{1}{2}} \times \left(-2 \times \frac{16384}{x^3} + \frac{x}{2}\right)$$

$$= \frac{-\frac{16384}{x^3} + \frac{x}{4}}{\sqrt{\left(\frac{16384}{x^2} + \frac{x^2}{4}\right)}}$$

At $\frac{dd}{dx} = 0$, $\frac{16384}{x^3} = \frac{x}{4}$

$x = 16$

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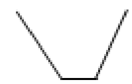
Using the sign test

x	0^+	16	20
$\frac{dd}{dx}$	–	0	+



Therefore minimum at $x = 16,\ y = 8$

23 $V = x^2y = 64\,000 \text{ cm}^3$

$SA = 2x^2 + 4xy$, but $y = \dfrac{64\,000}{x^2}$

$$SA = 2x^2 + 4x\left(\frac{64\,000}{x^2}\right)$$
$$= 2x^2 + \frac{256\,000}{x}$$

Minimum SA when $\dfrac{dSA}{dx} = 0$ and $\dfrac{d^2SA}{dx^2} > 0.$

$$\frac{dSA}{dx} = 4x - \frac{256\,000}{x^2}$$

$$\frac{d^2SA}{dx^2} = 4 + 2 \times \frac{256\,000}{x^3}$$

At $\dfrac{dSA}{dx} = 0,\quad 4x = \dfrac{256\,000}{x^2} \Rightarrow x = 40$

At $x = 40,\ \dfrac{d^2SA}{dx^2} > 0$, so minimum surface area.

$SA = 9600 \text{ cm}^2$

24 $P(n) = R(n) - C(n)$

$$P(n) = 50n - \left(n^2 - 6n + 35\right)n$$
$$= 50n - n^3 + 6n^2 - 35n$$
$$= -n^3 + 6n^2 + 15n$$

Maximum profit when $P'(n) = 0$ and $P''(n) < 0$.

$P'(n) = -3n^2 + 12n + 15$

$P''(n) = -6n + 12$

At $P'(n) = 0$, $-3n^2 + 12n + 15 = 0 \Rightarrow n^2 - 4n - 5 = 0$

$(n-5)(n+1) = 0$

$n > 0$; so $n = 5$

At $n = 5$, $P''(n) < 0$, so maximum profit.

25 $C(x) = 0.005x^2 - 2^x + 250$

a $P(x) = R(x) - C(x)$

$P(x) = dx - C(x)$

$$P(x) = x\left(300 - \frac{x}{2}\right) - (0.005x^2 - 2x + 250)$$
$$= 300x - \frac{x^2}{2} - 0.005x^2 + 2x - 250$$
$$= -0.505x^2 + 302x - 250$$

Maximum profit when $P'(x) = 0$ and $P''(x) < 0$.

$P'(x) = -1.01x + 302$

$P''(x) = -1.01$

At $P'(x) = 0$, $x = 299$

At $x = 299$, $P''(x) < 0$, so maximum profit.

For maximum profit, make 299 bears.

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b $d = 300 - \frac{x}{2} = \$150.50$

c $P(x) = -0.505x^2 + 302x - 250$

$P(299) = \$44\,900.50$

26 $h = -3t^2 + 6t + 1$ and $d = 2t$.

Maximum height when $h'(t) = 0$ and $h''(t) < 0$.
$h'(t) = -6t + 6$
$h''(t) = -6$
At $h'(t) = 0$, $t = 1$
At $t = 1$, $h''(t) < 0$, so maximum height.
At $t = 1$, $x = 4$ m

Maximum height is 4 m.

27 $P = -\frac{1}{3}t^3 + 16.5t^2 + 70t + 5000$

Maximum population when $P'(t) = 0$ and $P''(t) < 0$.
$P'(t) = -t^2 + 33t + 70$
$P''(t) = -2t + 33$
At $P'(t) = 0$, $t = -2$ or $t = 35$ but $t > 0$
At $t = 35$, $P''(t) < 0$, so maximum population.

Temperature that results in maximum population is 35°C.